Chapter VII

Dimension

One of the first indication that definable sets in o-minimal structures have good geometrical properties is that there is a good notion of dimension. This is not the case in general in topology: for instance, any non-empty compact connected and locally connected metric space Y is the image of a surjective continuous map

$$f : [0,1] \longrightarrow Y$$

whereas a standard requirement of "dimension" is that $\dim f(X) \leqslant \dim X$ for $f : X \longrightarrow Y$ "reasonable".

1. Definition(s)

As usual we fix an o-minimal structure M.

For any $c \in \{0,1\}^m = \mathbb{B}_m$, we define

$$\dim(c) = \sum_{i=1}^{m} c_i .$$

<u>Definition</u> - Let $m \geq 0$ and $X \subset M^m$ definable [with parameters] and not empty.

The dimension of X is

$$\dim(X) = \operatorname{Sup} \{ \dim(c) \mid \exists\, C \in \mathrm{Cells}_c(M) \text{ and } \varphi : C \hookrightarrow X \text{ definable and injective} \}$$

$[\in \mathbb{N} \cup \{+\infty\}$ a priori$]$

It is entirely clear that $\dim(X)$ is finite, but we will check it soon. However we get immediately

<u>Lemma</u> - (1) If $\emptyset \neq X \subset Y$ then $\dim(X) \leq \dim(Y)$.

(2) If $f : \underset{M^m}{X} \longrightarrow \underset{M^n}{Y}$ is $\left\{ \begin{matrix} \text{definable} \\ \text{and} \\ \text{injective} \end{matrix} \right\}$, then we have $\dim f(X) \leq \dim X$. In particular, f bijective $\Rightarrow \dim(X) = \dim(Y)$

(3) $\dim(X) = \operatorname{Sup} \{ m \mid \exists\, C \in \mathrm{Cells}_{1_m}(M)$ and $\exists\, \varphi : C \hookrightarrow X \}$

Proof. (1) is clear: any $\varphi : C \hookrightarrow X$ gives

$$C \xhookrightarrow{\varphi} X \subset Y \quad \text{injective.}$$

(2) similarly any $\varphi : C \hookrightarrow X$ gives

$C \xhookrightarrow{\varphi} X \xhookrightarrow{f} f(X)$, $f \circ \varphi$ injective so

$$\dim f(X) \geq \dim(X).$$

If f is bijective then we get $\dim(Y) \geq \dim(X)$ and then $\dim(Y) = \dim(X)$ by applying this to f^{-1} also.

(3): this is because there is a definable homeo-morphism $p : \underset{\mathrm{Cells}_s(M)}{\underset{\cap}{C}} \xrightarrow{\sim} \underset{\mathrm{Cells}_{1_{\dim(C)}}(M)}{\underset{\cap}{\tilde{C}}} \subset M^{\dim(C)}$.

$\square$

The following simple lemma will imply quite easily all properties of the dimension.

Lemma. Let $C \subset M^m$ be an open cell.

For any definable injective $f : C \to M^n$, the image $f(C)$ contains a cell $D \in \mathrm{Cells}_d(M)$

such that $\dim(d) \geq m$.

<u>Proof.</u> The statement is certainly true for $m=0$ so we assume $m \geq 1$ and argue by induction.

Let $f : C \longrightarrow M^n$ be injective and definable with $C \subset M^m$ open.

<u>Case 1.</u> We show that the case $n < m$ is impossible. Indeed, pick $x_0 \in M^{m-1-n}$; then consider the injective map

$$\begin{array}{ccccc} C & \hookrightarrow & M^n & \hookrightarrow & M^{m-1} \\ x & \longmapsto & f(x) & & \\ & & y & \longmapsto & (y, x_0) \end{array}$$

to reduce to the case $n = m-1$.

By cellular decomposition, we find an open cell $D \subset C$ s.t. $f|D$ is continuous. Now D contains a cell $\tilde{D}$ of type $(1, \dots, 1, 0)$ (easy) and by induction the image of $f|\tilde{D}$ contains an <u>open</u> cell in M^{m-1}; then the preimage of this is contained in $\tilde{D}$ (because f is injective), contradicting the fact that $f|D$ is continuous (so the preimage should be open, whereas $\tilde{D}$ has empty interior).

<u>Case 2.</u> Suppose that $n \geq m$. Let then $\mathcal{D}$ be a cellular decomposition of $f(C)$, and pick a cell $D \in \mathrm{Cells}_d(M)$ such that $f^{-1}(D)$ contains an open cell $\tilde{C} \subset C$; consider further

$$\tilde{C} \xhookrightarrow{f|\tilde{C}} D \xrightarrow{p} \tilde{D} \subset M^{\dim(d)}.$$

This composition is injective and definable, so by Case 1, we have $\dim(d) \geq m$.

$\square$

<u>Corollary 1.</u> Let $\emptyset \neq X \subset M^m$ be definable.

We have

$$\dim(X) = \underbrace{\sup\{\dim(c) \mid \exists C \in \mathrm{Cells}_c(M) \text{ s.t. } C \subset X\}}_{\leq m}$$

This is the def. in the book of van den Dries, which has the esthetical problem that it is not obviously invariant by def. homeo.

Proof. Let $\delta(X)$ be the RHS. We have

$$\dim(X) \geq \delta(X)$$

because any $C \subset X$ gives a $\varphi : C \hookrightarrow X$.

Conversely, if $\underset{M^n}{\overset{C}{\cap}} \overset{\varphi}{\hookrightarrow} X$ is definable and injective, and C open, then by the Lemma, $\varphi(C)$ contains a $D \in \mathrm{Cells}_d(M)$ with $\dim(d) \geq n$, so $\delta(X) \geq n$, hence $\delta(X) \geq \dim(X)$.

$\square$

Corollary 2. If $C \in \mathrm{Cells}_c(M)$, then $\dim(C) = \dim(c)$.

Proof. We may assume C open (so $C \in \mathrm{Cells}_{\underbrace{(1,\dots,1)}_{\dim(c)}}(M)$) and then the result is clear by the first corollary.

$\square$

Corollary 3. Let $\emptyset \neq X_i \subset M^m$ for $i \in I$, I finite.

Then $$\dim\left(\bigcup_{i \in I} X_i\right) = \max_{i \in I}(\dim X_i)$$

Proof. By induction, we may assume that $I = \{1, 2\}$.

Then $\dim(X_i) \leq \dim(X_1 \cup X_2)$ clearly.

Let then $C \subset X_1 \cup X_2$ be a cell of dimension $d = \dim(X_1 \cup X_2)$. Let $p_c : C \xrightarrow{\sim} \tilde{C} \subset M^d$ be the definable projection to an open cell.

Then one of $p(X_1 \cap C)$, $p(X_2 \cap C)$ must contain an open cell (since $\tilde{C} = p(X_1 \cup C) \cup p(X_2 \cup C)$)

say $p(X_i \cap C) \supset D$. Then

$$X_i \cap C \supset p^{-1}(D)$$

which is a cell of dimension $\dim(C)$ inside X_i, so $\dim(X_i) \geq d$, which gives the result.

$\square$

2 - Dimension and fibers

We now consider a common theme in many geometries: given $f : X \longrightarrow Y$ definable, what are the relations between $\dim(X)$, $\dim f(X)$ and $\dim f^{-1}(a)$ for varying $a \in Y$?

As in algebraic geometry, we begin with what is apparently a very special case.

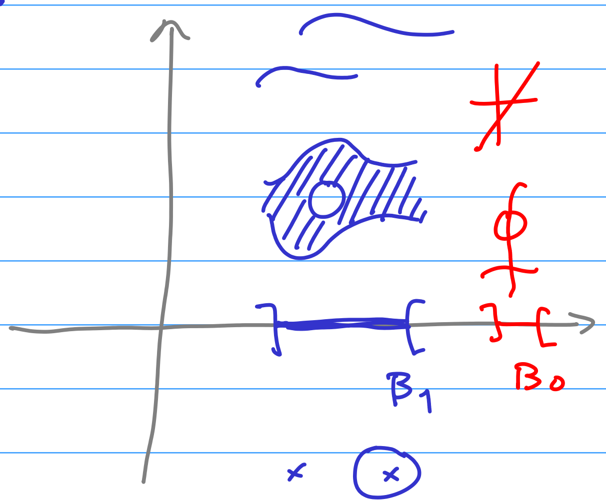
<u>Proposition</u>.

$$\left.\begin{array}{c} X \subset M^m \\ \big\downarrow{\scriptstyle p_1} \\ B \subset M^n \\ \shortparallel \\ p_1(X) \end{array}\right\} \quad \begin{array}{l} \text{fibers} \\ \tilde{X}_a \subset M^{m-n} \\ \quad \wr\!\shortparallel \\ X_a \end{array}$$

(a) $B_k = \{ a \in B \mid \dim(X_a) = k \}$ is definable for any $k \in \mathbb{N}$

(b) For any $k \in \mathbb{N}$, we have

$$\dim(B_k) + k = \dim(\underbrace{X_{B_k}}_{\{(a,b) \in M^n \times M^{m-n} \mid a \in B_k\}})$$



<u>Proof</u>. Let $\mathcal{D}$ be a cellular decomposition of X. The projections $\pi_{m,n}(C)$ for $C \in \mathcal{D}$ form, by definition, a cellular decomposition $\tilde{\mathcal{D}}$

of B. Pick a cell $\tilde{C}$ of $\tilde{\mathcal{D}}$; then, again by definition, the space $X_{\tilde{C}}$ over $\tilde{C}$ is the (disjoint) union of the cells $C \in \mathcal{D}$ s.t.

$$\pi_{m,n}(C) = \tilde{C}.$$

Let $\tilde{\mathcal{D}}_{\tilde{C}}$ be the set of these cells.

Then we get (by Cor. 3 above)

$$\dim(X_{\tilde{C}}) = \sup_{C \in \tilde{\mathcal{D}}_{\tilde{C}}} \dim(C)$$

and for all $a \in \tilde{C}$

$$\dim(X_a) = \sup_{C \in \tilde{\mathcal{D}}_{\tilde{C}}} \dim(C_a).$$

The key fact is that, in addition, we have

$$\dim(C) = \dim(\tilde{C}) + \dim(C_a)$$

if $C \in \tilde{\mathcal{D}}_{\tilde{C}}$ and $a \in \tilde{C}$. The reason is that if

C is of type $(c_1, \dots, c_m)$, then

$\tilde{C}$ ———— $(c_1, \dots, c_n)$ and

C_a ———— $(c_{n+1}, \dots, c_m)$.

more properly the image in M^{m-n}

So we get $\dim(X_a) = \sup_{C \in \tilde{\mathcal{D}}_{\tilde{C}}} \left(\dim(C) - \dim(\tilde{C})\right)$ for all $a \in \tilde{C}$. This means that the dimension of X_a is constant on cells of $\tilde{\mathcal{D}}$, so the set of a where X_a has a given dimension is a union of cells of $\tilde{\mathcal{D}}$, so is definable.

This proves (a). To deduce (b), we note that B_k is the union of cells $\tilde{C} \in \tilde{\mathcal{D}}$ s.t.

$$(*) \qquad k = \sup_{C \in \tilde{\mathcal{D}}_{\tilde{C}}} \left(\dim(C)\right) - \dim \tilde{C}$$

and X_{B_k} is the union of the corresponding $X_{\tilde{C}}$.

So

$$\begin{aligned}
\dim(X_{B_k}) &= \sup_{\tilde{C},\,(*)} \dim(X_{\tilde{C}}) \\
&= \sup_{\tilde{C},(*)} \ \sup_{C \in \tilde{\mathcal{D}}_{\tilde{C}}} \dim(C) \\
&= \sup_{\tilde{C},(*)} \left(k + \dim(\tilde{C})\right) \\
&= k + \dim B_k .
\end{aligned}$$

$\square$

Corollary. $f: \underset{M^m}{\overset{\cup}{X}} \longrightarrow \underset{M^n}{\overset{\cup}{Y}}$ definable.

(a) For all $k \in \mathbb{N}$, $Y_k = \{ y \in Y \mid \dim f^{-1}(y) = k \}$ is definable.

(b) If $Y_k \neq \emptyset$ then $\dim f^{-1}(Y_k) = k + \dim(Y_k)$.

(c) $\dim f(X) \leq \dim(X)$.

Proof. Consider $\tilde{\Gamma}_f = \{ (f(x), x) \mid x \in X \} \subset M^{n+m}$ and the diagram

$$\begin{tikzcd} X \arrow[dr, "f"'] & \tilde{\Gamma}_f \arrow[l, "p_2"', "\sim"] \arrow[d, "p_1"] \\ & f(X) \subset Y \end{tikzcd}$$

which is commutative. So for any $y \in f(X)$, we have $f^{-1}(y) \xrightarrow[p_2^{-1}]{\sim} p_1^{-1}(y)$ (definable bijection)

so $\begin{cases} Y_k = \{ y \mid \dim (\tilde{\Gamma}_f)_y = k \} \\ (\tilde{\Gamma}_f)_{Y_k} \xrightarrow[p_2]{\sim} f^{-1}(Y_k) \end{cases}$

and hence (a), (b) follow from the proposition.

For (c) note that

$$\dim(f(X)) = \sup_k \dim(Y_k)$$

(since only finitely many Y_k are $\neq \emptyset$) so

$$\dim f(X) \leq \sup_k \left(\dim \left(f^{-1}(Y_k) \right) - k \right)$$

$$\leq \sup_k \dim f^{-1}(Y_k) \leq \dim X.$$

$\square$