Sheet 1

Exercise 1

Let K be a fixed field. Define a language $\mathcal{L}$ such that K-vector spaces are naturally $\mathcal{L}$-structures.

Exercise 2

Let $\mathcal{L}_{\mathrm{r}} = (+, -, \cdot, 0, 1)$ be the language of rings.

(a) Explain why the notion of isomorphic rings (considered as $\mathcal{L}_{\mathrm{r}}$-structures in the obvious way) corresponds to that of isomorphism in the usual algebraic sense.

(b) Show that if we consider a field as an $\mathcal{L}_{\mathrm{r}}$-structure, the substructures do not always coincide with the subfields.

Exercise 3

In the language $(\cdot, e)$ of groups, show that there exists a sentence ϕ such that $\mathcal{M} \models \phi$ if and only if $\mathcal{M}$ if isomorphic to $\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$.

Exercise 4

Find an *axiomatization* of integral domains in the language of rings, i.e., a set $\mathcal{A}$ of sentences in the language of rings such that a ring A is an integral domain if and only if $A \models \phi$ for all $\phi \in \mathcal{A}$.

Exercise 5

Find a language $\mathcal{L}$ and a sentence ϕ in $\mathcal{L}$ such that the set of cardinalities of the finite $\mathcal{L}$-structures that satisfy ϕ coincides with the set of powers of primes (excluding $p^0 = 1$).

Exercise 6

In the language $\mathcal{L} = (+, 0)$ (with $+$ a binary function and 0 a constant), show that there exists a sentence ϕ such that $\mathbb{Z} \times \mathbb{Z} \models \phi$ but $\mathbb{Z} \not\models \phi$ (One says that these two structures are not *elementarily equivalent*).

Exercise 7

Let $\mathcal{L}_{\mathrm{r}} = (+, -, \cdot, 0, 1)$ be the language of rings. Let $\mathcal{M} = \mathbb{C}(T)$ viewed as an $\mathcal{L}_{\mathrm{r}}$-structure in the obvious way.

1. Show that there do not exist non-constant rational functions f, g in $\mathbb{C}(T)$ such that $g^2 = f^3 + 1$.

2. Deduce that the formula

$$\phi(v)\colon \exists\, x\, \exists\, y\, (y^2 = v \wedge x^3 + 1 = v)$$

has the property that $\phi(\mathbb{C}(T)) = \mathbb{C}$ (so the field of constants is definable in $\mathbb{C}(T)$.