Sheet 2

Exercise 1

Let $\mathcal{L} = \{(E, 2)\}$ be the language of graphs containing a single binary relation E. Prove that there is no $\mathcal{L}$-theory T whose models are exactly the trees.

Exercise 2

Let $\mathcal{L}$ be a language and T an $\mathcal{L}$-theory. Let $\varphi(x, y)$ be a formula with two variables. Assume that for every model M of T and every $b \in M$, the set

$$\varphi(M, b) = \{a \in M \mid M \models \varphi(a, b)\}$$

is finite.

Prove that there is an integer $C \geq 0$ such that all the sets $\varphi(M, b)$ have cardinality at most C, as M ranges over models of T and b ranges over M.

Hint: assume this is false, and then expand the language with infinitely many new constant symbols (c_i), and an additional constant t, and expand the theory in a suitable way, so that after showing that it is finitely-satisfiable, a contradiction follows.

Exercise 3

1. Let E be a finite field and let $\bar{E}$ be an algebraic closure of E. Let $n \geq 1$ be an integer and let $f_1, \ldots, f_n$ be polynomials in $\bar{E}[X_1, \ldots, X_n]$. Assume that the map $x = (x_i) \mapsto (f_j(x))$ is injective. Prove that this map is also surjective.

2. Give an example to show that "injective" and "surjective" cannot be switched.

3. Show that an ultrafilter is principal if and only if it contains a finite set.

4. Let C be a non-principal ultraproduct of fields E_p which are algebraic closures of $\mathbb{F}_p$ as p ranges over all prime numbers. Show that C is an algebraically-closed field of characteristic 0.

5. Show that the cardinality of C is bounded by that of $\mathbb{C}$.

6. Show that there exists a family $(f_t)_{t \in \mathbb{R}}$ of maps $f_t \colon \mathbb{N} \to \mathbb{Q}$ such that, for all $t \neq s \in \mathbb{R}$, the set

   $$\{n \geq 0 \mid f_t(n) = f_s(n)\}$$

   is finite

   Hint: consider for each t a sequence of rational numbers converging to t.

7. Deduce the existence of a family of maps g_t from the set of primes to the disjoint union of all E_p such that $g_t(p) \in E_p$ for all primes p and for all $t \neq s \in \mathbb{R}$, the set

   $$\{p \geq 0 \mid g_t(p) = g_s(p)\}$$

   is finite.

8. Deduce that the cardinality of C is equal to that of $\mathbb{C}$, and conclude that C is isomorphic to $\mathbb{C}$ as a field. (Use the fact that algebraically closed fields of characteristic 0 are isomorphic if and only if they have the same cardinality.)

9. Let $g_1, \ldots, g_n$ be elements of $\mathbb{C}[X_1, \ldots, X_n]$. If the map $z = (z_i) \mapsto (g_j(z))$ is injective from $\mathbb{C}^n$ to itself, then it is surjective.