Sheet 4

Exercise 1

Let $M = X^2Y^2(X^2 + Y^2 - 3) + 1 \in \mathbb{R}[X, Y]$. We saw in class, that every non-negative polynomial is a sum of squares of rational functions (Hilbert's 17th problem). In this exercise, we will see that M is a non-negative polynomial that is not a sum of squares of polynomials.

(a) Show that $M(x, y) \geq 0$ for all $(x, y) \in \mathbb{R}^2$.

Hint: Use the arithmetic-geometric mean inequality on three variables.

(b) Show that if $m \geq 1$ and $(f_1, \ldots, f_m)$ are non-zero elements of $\mathbb{R}[X, Y]$, then

$$\deg(f_1^2 + \cdots + f_m^2) = 2 \max(\deg(f_i)).$$

(c) Show that there is no finite family $(p_i)_{i \in I}$ in $\mathbb{R}[X, Y]$ such that $M = \sum p_i^2$.

Hint: Assume that there is such a family; show first that $\deg(p_i) \leq 3$*, then evaluate with* $X = 0$ *and* $Y = 0$ *to see that each* p_i *would have to be of the form* $a + bXY$ *for some* $a \in \mathbb{R}$ *and some* $b \in \mathbb{R}[X, Y]$ *with degree at most* 1*; compute then the coefficient of* $(XY)^2$.

Exercise 2

Let $\mathcal{L}$ be a language with a binary relation symbol $\leq$.

(a) Let M be an o-minimal $\mathcal{L}$-structure. Show that a non-empty subset $X \subset M$ with $X \neq M$ is definable if and only if the boundary $\partial X = \overline{X} \setminus \mathring{X}$ of X is finite and non-empty.

(b) Let M be an $\mathcal{L}$-structure in which $\leq$ is interpreted as a total order which is dense without endpoints. Show that M is o-minimal if and only if every definable non-empty subset $X \subset M$ with $X \neq M$ has finite non-empty boundary $\partial X = \overline{X} \setminus \mathring{X}$, and for any $x < y$ in $\partial X \cup \{-\infty, +\infty\}$, if $]x, y[\cap \partial X$ is empty, then either $]x, y[\subset X$ or $]x, y[\cap X = \emptyset$.

Exercise 3

Let $\mathcal{L}_0 = (\cdot, e, \leq)$ be the language of ordered groups. Let M be an o-minimal $\mathcal{L}_0$-structure which is a model of the theory of ordered groups. (Which means that the order has the property that $x \leq y$ implies $xz \leq yz$ and $zx \leq zy$ for all $z \in M$.)

(a) Let $H \subset M$ be a definable subgroup of $(M, \cdot)$. Show that H is an interval, i.e., if $e < h$ for some $h \in H$, then $[e, h] \subset H$.

Hint: By contradiction, show that if this is false, then there is an infinite "discrete" definable set.

(b) Show that the only definable subgroups of $(M, \cdot)$ are $\{e\}$ and M.

(c) Deduce that $(M, \cdot)$ is abelian and divisible, i.e. that for any $y \in M$ and $n \geq 1$ integer, there exists $x \in M$ such that $x^n = y$.

Exercise 4

Let $\mathcal{L} = (+, -, \cdot, 0, 1, \leq)$ be the language of ordered rings. Let M be an o-minimal $\mathcal{L}$-structure which is a model of the theory of ordered rings (not necessarily commutative; this means that $0 < 1$ in M, that $(M, +)$ is an ordered abelian group, and that the order has the property that whenever $x \leq y$ and $z \geq 0$, also $xz \leq yz$). *Hint: Use exercise 3.*

(a) Show that for every $x \in M \setminus \{0\}$, there is an inverse element $y \in M$ with $xy = 1$.

(b) Show that the positive elements of M form an ordered group with the multiplication.

(c) Show that M is an ordered field.

(d) Show that positive elements in M have a square root.

(e) Show that addition and multiplication are continuous on M^2 (with the order topology on M and the product of the order topology on M^2).

(f) Show that for $f \in M[X]$, the polynomial function associated to f from M to M is a definable continuous function.

(g) Show that M is a real-closed field. *Hint: Use the criterion that a field F is real closed if and only if (1) for every $a \in F$, either a or $-a$ is a square and (2) every polynomial in $F[X]$ of odd degree has a root in F.*