Sheet 5

Let $\mathcal{L}$ be a language containing $\leq$ and let M be an o-minimal $\mathcal{L}$-structure. We say "definable" for "definable with parameters".

Exercise 1

(a) Let C be a cell of type $c = (c_i)_{1\leq i\leq m}$. Show that C is open in M^m if and only if $c_i = 1$ for all i.

(b) Show that if I_i is a non-empty open interval for $1 \leq i \leq m$, then $I_1 \times \cdots \times I_m$ is an open cell in M^m.

Exercise 2

The goal of this exercise is to show that if $X \subset M^m$ is a definable set which is the union of finitely many cells C_i and if X has non empty interior, then one of the C_i is an open cell (without using cell decompositions!).

We do this by induction on m, so assume that the property holds for subsets of M^{m-1}. Assume that the interior of $X \subset M^m$ is not empty, and let $x_0 \in X$ be an interior point. Let $x_0' \in M^{m-1}$ be the projection on the first $m-1$ coordinates, and similarly use A' to denote the image of a subset $A \subset M^m$.

(a) Show that there exists an open neighborhood U of x_0' such that the fiber X_y of the projection $X \to M^{m-1}$ is infinite if $y \in U$.

(b) Show that there exists $y \in U$ such that X_y is contained in the union of the C_i where C_i' is open in M^{m-1}.

(c) Conclude that some C_i is an open cell.

Exercise 3

We assume that $\mathcal{L}$ extends the language of ordered rings, so that M is an ordered ring. For any $n < m$, let $\pi_{m,n} \colon M^m \to M^n$ be the projection to the first n coordinates.

(a) For any cell $C \subset M^m$, show that there is a definable homeomorphism $C \to M^{\dim(C)}$.

(b) Let $X \subset M^m$ be definable. show that there is a definable map $\sigma \colon \pi_{m,n}(X) \to X$ such that $\pi_{m,n} \circ \sigma$ is the identity.

Exercise 4

For a cell C, we define $\chi(C) = (-1)^{\dim(C)}$. For a finite family $\mathcal{C} = (C_i)_{i\in I}$ of disjoint cells in M^m, we define

$$\chi(\mathcal{C}) = \sum_{i\in I} \chi(C_i) = \sum_{k=0}^{m} (-1)^k n_k$$

where n_k is the number of cells of dimension k in $\mathcal{C}$.

(a) Let $\mathcal{D}$ be a cellular decomposition of a cell $C \subset M^m$. Show that

$$\chi(\mathcal{D}) = \chi(C).$$

(Hint: use induction on m, and sum over the projections of the cells in M^{m-1}, according to their type.)

(b) Let $X \subset M^m$ be definable. Show that $\chi(\mathcal{D})$ is independent of $\mathcal{D}$ for all cellular decompositions of X. This common value is denoted $\chi(X)$.

(c) Show that if X_1 and X_2 are definable subsets of M^m, then

$$\chi(X_1 \cup X_2) = \chi(X_1) + \chi(X_2) - \chi(X_1 \cap X_2).$$

(Hint: first treat the case where X_1 and X_2 are disjoint.)

(d) Let $X \subset M^m$ be definable and let $n < m$. Show that for any $k \in \mathbb{Z}$, the set

$$\{a \in M^n \mid \chi(X_a) = k\}$$

is definable.