Sheet 6

Let $\mathcal{L}$ be a language containing $\leq$ and let M be an o-minimal $\mathcal{L}$-structure. We say "definable" for "definable with parameters". This sheet is about point-set topology of definable sets.

Exercise 1

The goal of this exercise is to prove that definable sets in $\mathbb{R}^m$ are connected if and only if they are d-connected. Suppose that $M = \mathbb{R}$ with the usual interpretation of the order.

(a) Let $X \subseteq \mathbb{R}^m$ be a definable set which is connected *in the usual topological sense*. Show that X is d-connected.

(b) Let $C \subseteq \mathbb{R}^m$ be a cell. Show that C is connected in the usual topological sense.

(c) Let $X \subseteq \mathbb{R}^m$ be a d-connected definable set and $\mathcal{D}$ a cellular decomposition of X. Assume that $X = U \cup V$ where U and V are disjoint open sets in X (for the usual topology, so $U = U_1 \cap X$ where $U_1 \subset \mathbb{R}^m$ is open, etc). Show that for any cell C of $\mathcal{D}$, we have $C \subset U$ or $C \subset V$.

(d) Deduce that U and V are adapted to $\mathcal{D}$ and conclude that d-connected definable subsets $X \subseteq \mathbb{R}^m$ are connected in the usual sense.

(e) Find an o-minimal structure M and a d-connected definable subset which is not connected in the usual topology.

Exercise 2

We assume that $\mathcal{L}$ extends the language of ordered rings, so that we know that M is a real closed (ordered) field. Let $m \geq 1$ be an integer. We denote $|x| = \max(x, -x)$ for $x \in M$, and we put $\|x\| = \max(|x_i|)$ for $x = (x_i) \in M^m$. A subset $X \subset M^m$ is *bounded* if and only if there exists $A \in M$ such that $\|x\| \leq A$ for all $x \in A$.

(a) Prove that the topology of M^m is generated by the sets of the form $\{x \in M^m \mid \|x - x_0\| < \delta\}$ for $x_0 \in M^m$ and $\delta > 0$ in M.

(b) Let $X \subset M^m$ be definable and let x_0 be an element of M^m belonging to the closure of X. This means that there is a sequence $x_n \in X$ with $\lim_{n\to\infty} x_n = x_0$. Sequences are not definable, but their role can be replaced by definable maps $\gamma\colon]0, c[\to X$ with $\lim_{x\to 0} \gamma(x) = x_0$.

  (i) Assume that $x_0 \in \overline{X} \setminus X$. Prove that there exists a non-empty open interval $I =]0, c[$ and a definable map $\gamma\colon I \to X$ such that $\|x_0 - \gamma(t)\| = t$ for all $t \in I$.
  Hint: use Exercise 3 of Exercise Sheet 5.

  (ii) Prove that there exists $c > 0$ in M and a continuous definable map $\gamma\colon]0, c[\to X$ such that $\lim_{t\to 0} \gamma(t) = x_0$.
  Hint: consider separately the case when $x_0 \in X$.

(c) Let $C \subset M^m$ be a *bounded* cell and $\overline{C}$ its closure. Let $\pi\colon M^m \to M^{m-1}$ be the projection that omits the last coordinate. Show that $\overline{\pi(C)} = \pi(\overline{C})$.

Hint: deal separately with the cases where C is a graph or the space between two graphs of continuous definable functions on $\pi(C)$; apply the previous exercise to show that if $a \in M^{m-1}$ is in the closure of $\pi(C)$, then it is in $\pi(\overline{C})$.

(d) Let $X \subset M^m$ be closed, bounded and definable. Let $f\colon X \to M^n$ be definable and continuous. The goal of this exercise is to prove that $f(X)$ is bounded.

  (i) Assuming that $f(X)$ is not bounded, show that there exists a definable map $g\colon M \to X$ such that $\|f(g(t))\| > t$ for all $t \in M$.

  (ii) Show that the limit of $g(t)$ as $t \to +\infty$ in $\overline{X}$ exists.
  Hint: apply the monotonicity theorem to each coordinate of g.

  (iii) Deduce a contradiction, hence the result.