Solutions to sheet 2

Exercise 1

Let $\mathcal{L} = \{(E, 2)\}$ be the language of graphs containing a single binary relation E. Prove that there is no $\mathcal{L}$-theory T whose models are exactly the trees.

Solution:
Graphs are $\mathcal{L}$-structures by considering the vertices as elements of the universe and by having the relation $E(x, y)$ between two vertices x, y if and only if there is an edge between x, y. We assume for contradiction that there is an $\mathcal{L}$-theory T, whose models are exactly trees. We introduce a new Language $\mathcal{L}' = \mathcal{L} \cup \{a, b\}$, where a and b are constant-symbols. For $n \in \mathbb{N}$, we then consider the $\mathcal{L}'$-sentences

$$\phi_n \quad = \quad \neg\exists x_1 \dots x_n, x_1 = a \wedge x_n = b \wedge \bigwedge_{i=1}^{n-1} E(x_i, x_{i+1}) \vee x_i = x_{i+1}$$

which intuitively say that there is no path of length at most n between a and b. We consider the $\mathcal{L}'$-theory $T' = T \cup \{\phi_n : n \in \mathbb{N}\}$ and show that it is finitely satifyable. Indeed, let $\Delta \subseteq T'$ be a finite subset. Let $N = \max\{n \in \mathbb{N} : \phi_n \in \Delta\}$. We construct a $\mathcal{L}'$-model (the infinite linear graph) $\mathcal{M}$ satisfying Δ: Let $\mathbb{Z}$ be the universe and $E_{\mathcal{M}}(z_1, z_2)$ if and only if $|z_1 - z_2| = 1$. We may choose $a_{\mathcal{M}} = 0$ and $b_{\mathcal{M}} = N + 1$. Since $\mathcal{M}$ is a tree, it satisfies $\mathcal{M} \models T$ and by the choice of $a_{\mathcal{M}}, b_{\mathcal{M}}$ $\mathcal{M} \models \phi_n$ for $n < N$, hence $\mathcal{M} \models \Delta$. By the compactness-theorem, there exists a model $\mathcal{N}$ of T'. Since $\mathcal{N} \models T$, $\mathcal{N}$ has to be a tree (with marked points $a_{\mathcal{N}}$ and $b_{\mathcal{N}}$). But since $\mathcal{N} \models \phi_n$ for all $n \in \mathbb{N}$, it is not possible to connect $a_{\mathcal{N}}$ and $b_{\mathcal{N}}$ by a path, hence $\mathcal{N}$ is not connected. This is a contradiction, and hence there is no theory T that axiomatizes trees in the language $\mathcal{L}$.

Exercise 2

Let $\mathcal{L}$ be a language and T an $\mathcal{L}$-theory. Let $\varphi(x, y)$ be a formula with two variables. Assume that for every model M of T and every $b \in M$, the set

$$\varphi(M, b) = \{a \in M \mid M \models \varphi(a, b)\}$$

is finite.

Prove that there is an integer $C \geq 0$ such that all the sets $\varphi(M, b)$ have cardinality at most C, as M ranges over models of T and b ranges over M.

Hint: assume this is false, and then expand the language with infinitely many new constant symbols (c_i), and an additional constant t, and expand the theory in a suitable way, so that after showing that it is finitely-satisfiable, a contradiction follows.

Solution:
We assume for contradiction that for every $n \in \mathbb{N}$ there is a model M_n of

T and an element $b_n \in M_n$ such that $|\varphi(M_n, b_n)| \geq n$. We define a new language $\mathcal{L}' = \mathcal{L} \cup \{c_i : i \in \mathbb{N}\} \cup \{t\}$, where the c_i and the t are constant-symbols. We consider the $\mathcal{L}'$-sentences

$$\psi_n = \bigwedge_{i,j=1, i\neq j}^{n} c_i \neq c_j \wedge \bigwedge_{i=1}^{n} \varphi(c_i, t)$$

which intuitively say that $c_1, \dots, c_n$ are distinct elements of $\varphi(M, t)$. We now consider the $\mathcal{L}'$-theory $T' = T \cup \{\psi_n : n \in \mathbb{N}\}$. We show that T' is finitely satisfyable: Let $\Delta \subseteq T'$ be a finite subset and let $N = \max\{n \in \mathbb{N} : \psi_n \in \Delta\}$. Then we construct a model $\mathcal{M}$ by extending M_N by choosing distinct $(c_i)_{\mathcal{M}} \in \varphi(M_N, b_N)$ (which exist by assumption) and $t_{\mathcal{M}} = b_N$. Note that $\mathcal{M} \models T$, since $M_N \models T$ and that $\mathcal{M} \models \Delta$ by our choice of $(c_i)_{\mathcal{M}}$ and $t_{\mathcal{M}}$. By the compactness theorem, it follows that there is a model $\mathcal{N} \models T'$. Since $\mathcal{N} \models T$, we know that $\varphi(\mathcal{N}, t_{\mathcal{N}})$ is finite. But if $N = |\varphi(\mathcal{N}, t_{\mathcal{N}})| \in \mathbb{N}$, we know that $\mathcal{N} \models \psi_{N+1}$, hence there are $N+1$ distinct elements $(c_i)_{\mathcal{N}} \in \mathcal{N}$ that all lie in $\varphi(\mathcal{N}, t_{\mathcal{N}})$. This is a contradiction, and hence our assumption must be false, i.e. the cardinalities of $\varphi(M, b)$ have to be bounded by a uniform constant $C \in \mathbb{N}$.

Exercise 3

1. Let E be a finite field and let $\bar{E}$ be an algebraic closure of E. Let $n \geq 1$ be an integer and let $f_1, \dots, f_n$ be polynomials in $\bar{E}[X_1, \dots, X_n]$. Assume that the map $x = (x_i) \mapsto (f_j(x))$ is injective. Prove that this map is also surjective.

Solution:
Note that in a finite field E, any injective map $E^n \to E^n$ is also surjective. Unfortunately, the algebraic closure $\bar{E}$ of a finite field is not finite, but we can fix this as follows: Consider the coefficients $a_i^j \in \bar{E}$ that appear in the polynomials

$$f_i(X) = \sum_{j=1}^{m} a_i^j X^j.$$

We assume that the map $\bar{E}^n \to \bar{E}^n$ given by the polynomials f_i is injective. We have to show that it is surjective. So consider $b = (b_1, \dots, b_n) \in \bar{E}^n$. We have to find a preimage of b.

Consider the algebraic field-extension $F = E[a_i^j, b_i]$ generated by the elements $a_i^j, b_i \in \bar{E}$. The field F is finite and the polynomials define an injective map $F^n \to F^n$, hence there exists a preimage $a = (a_1, \dots, a_n) \in F^n$, that gets sent to b. Hence both the maps $F^n \to F^n$ and $\bar{E}^n \to \bar{E}^n$ are surjective.

2. Give an example to show that "injective" and "surjective" cannot be swit-

ched.

Solution:
The map $\bar{E} \to \bar{E}$ defined by $x \mapsto x^2 - x$ is surjective since $\bar{E}$ is algebraically closed. But it is not injective since both 0 and 1 get sent to 0 in every field.

3. Show that an ultrafilter is principal if and only if it contains a finite set.

Solution:
Let X be a set and $\mathcal{F} \subseteq \mathcal{P}(X)$ an ultrafilter on X. If $\mathcal{F}$ is principal, then by definition there exists $x \in X$ with $\mathcal{F} = \{A \colon x \in A\}$, in particular $\{x\} \in \mathcal{F}$, hence $\mathcal{F}$ contains a finite set.

Let on the other hand $A \in \mathcal{F}$ be a finite subset of X. Without loss of generality, we may assume $|A| = \min\{|B| \colon B \in \mathcal{F}\} > 0$, ($B \neq \emptyset$ by (F1)). Let $x \in A$. Since $\mathcal{F}$ is an ultrafilter, we know that either $\{x\}$ or $X \setminus \{x\}$ is in $\mathcal{F}$. If $\{x\} \in \mathcal{F}$ we are done. If $X \setminus \{x\} \in \mathcal{F}$, then by (F2) also $A \cap (X \setminus \{x\}) = A \setminus \{x\} \in \mathcal{F}$, which contradicts minimality of $|A|$ and thus this case does not happen.

4. Let C be a non-principal ultraproduct of fields E_p which are algebraic closures of $\mathbb{F}_p$ as p ranges over all prime numbers. Show that C is an algebraically-closed field of characteristic 0.

Solution:
We use Łos' theorem about non-principal ultraproducts to prove the statements. Let $\mathcal{L}$ be the language of rings. The fields E_p as well as the non-principal ultraproduct C are $\mathcal{L}$-strucutres. We can encode the axioms of fields as $\mathcal{L}$-sentences and since they hold in E_p for all p, they also hold in C (by Łos). For being algebraically closed, a similar trick works, consider the $\mathcal{L}$-sentences

$$\varphi_n \quad = \quad \forall a_0, \ldots, a_n \, \exists x, a_n x^n + a_{n-1}x^{n-1} + \ldots + a_1 x + a_0 = 0$$

for $n \in \mathbb{N}$ and note that they hold for E_p and thus also for C (by Łos). To show that C has characteristic 0, we use

$$\psi_n \quad = \quad \neg \underbrace{1 + 1 + \cdots + 1}_{n-\text{times}} = 0$$

and notice that $E_p \models \psi_n$ for all but one p. Since we take the product over a non-principal ultrafilter $\mathcal{F}$, the set $\{p \colon E_p \models \psi_n\} = \{n\} \notin \mathcal{F}$ (by def of non-principal or by part 3) and hence by Łos, $C \not\models \psi_n$ for all $n \in \mathbb{N}$. Since the field C has no finite characteristic, it must have characteristic 0.

5. Show that the cardinality of C is bounded by that of $\mathbb{C}$.

Solution:
The ultraproduct C is a quotient of the countable product $\prod_p E_p$ of countable sets and hence has cardinality at most $\aleph_0^{\aleph_0} = \aleph_1 \leq |\mathbb{C}|$.

6. Show that there exists a family $(f_t)_{t\in\mathbb{R}}$ of maps $f_t \colon \mathbb{N} \to \mathbb{Q}$ such that, for all $t \neq s \in \mathbb{R}$, the set

$$\{n \geq 0 \mid f_t(n) = f_s(n)\}$$

is finite

Hint: consider for each t a sequence of rational numbers converging to t.

Solution:
For $t \in \mathbb{R}$ consider a sequence of rational numbers $(f_t(1), f_t(2), f_t(3), \ldots)$ converging to $t \in \mathbb{R}$. The function $f_t \colon \mathbb{N} \to \mathbb{Q}$ satisfies for $t \neq s$,

$$|\{n \geq 0 \mid f_t(n) = f_s(n)\}| < \infty$$

which is what we required.

7. Deduce the existence of a family of maps g_t from the set of primes to the disjoint union of all E_p such that $g_t(p) \in E_p$ for all primes p and for all $t \neq s \in \mathbb{R}$, the set

$$\{p \geq 0 \mid g_t(p) = g_s(p)\}$$

is finite.

Solution:
The fields E_p are countable, so let $\nu_p \colon \mathbb{Q} \to E_p$ be a bijection. We then define

$$g_t(p) := \nu_p(f_t(p))$$

using f_t from part 6. Clearly $g_t(p) \in E_p$ and the set

$$\{p \geq 0 \mid g_t(p) = g_s(p)\} = \{p \geq 0 \colon f_t(p) = f_s(p)\}$$

is finite for $s \neq t$ (as in part 6).

8. Deduce that the cardinality of C is equal to that of $\mathbb{C}$, and conclude that C is isomorphic to $\mathbb{C}$ as a field. (Use the fact that algebraically closed fields of characteristic 0 are isomorphic if and only if they have the same cardinality.)

Solution:
We define a function f by

$$\begin{aligned} \mathbb{R} \to \prod_p E_p \twoheadrightarrow \hat{\prod_{\mathcal{F}}} E_p \\ t \mapsto (g_t(p))_p \mapsto f(t). \end{aligned}$$

We want to show that f is injective. By the definition of the ultraproduct we have $f(t) = f(s)$ if and only if $\{p \geq 1 \colon g_t(p) = g_s(p)\} \in \mathcal{F}$. But this set is finite for $s \neq t$ by part 7 and a non-principal ultrafilter does not contain finite sets by part 3. Hence $f(s) \neq f(t)$ and f is injective. We conclude that $|\,\mathbb{R}\,| \leq |C|$ and together with part 5 that $|\,\mathbb{R}\,| = |C| = |\mathbb{C}|$. A general fact from algebra tells us now that algebraically closed fields of characteristic 0 and of the same cardinality are isomorphic. Hence C is isomorphic to $\mathbb{C}$.

9. Let $g_1, \ldots, g_n$ be elements of $\mathbb{C}[X_1, \ldots, X_n]$. If the map $z = (z_i) \mapsto (g_j(z))$ is injective from $\mathbb{C}^n$ to itself, then it is surjective.

Solution:
We can write this statement as first-order sentences:

$$\begin{aligned} \varphi_{n,m} \quad = \quad & \forall a_1^1, \ldots, a_n^m, \left(\forall xy, \bigwedge_{i=1}^{n} \sum_{j=1}^{m} a_i^j x^j = \sum_{j=1}^{m} a_i^j y^j \to x = y \right) \\ & \to \left(\forall y_1, \ldots, y_n \exists x_1, \ldots, x_n, \bigwedge_{i=1}^{n} \sum_{j=1}^{m} a_i^j x_i^j = y_i \right) \end{aligned}$$

and we have $E_p \models \varphi_{n,m}$ for all $n, m \in \mathbb{N}$ by part 1. By Łos, also $C \models \varphi_{n,m}$ and hence $\mathbb{C} \models \varphi_{n,m}$ by part 8. Thus for all $m \in \mathbb{N}$ and for all polynomials (of degree at most m), if the map defined from $\mathbb{C}^n$ to itself is injective, then it is surjective.

Note that the existence of a non-principal ultrafilter, the bijection $\mathbb{Q} \to E_p$ and the isomorphism $C \to \mathbb{C}$ are all non-canonical choices. Thus this proof is highly non-constructive.