Solutions to Sheet 5

Let $\mathcal{L}$ be a language containing $\leq$ and let M be an o-minimal $\mathcal{L}$-structure. We say "definable" for "definable with parameters".

Exercise 1

(a) Let C be a cell of type $c = (c_i)_{1\leq i\leq m}$. Show that C is open in M^m if and only if $c_i = 1$ for all i.

Solution:
We do indiction on m.

Let first C be open in M^m. If $m = 1$, C has to be a point or an interval and since points are not open it has to be an open interval, hence of type (1). Now assume that all open cells in M^{m-1} are of type $(1)_{i=1,\dots,m-1}$. Since C is open, we can find an open box $B = \prod]a_i, b_i[$ arounde every point $p \in B \subseteq C$. Projecting down one dimension, we know that $\pi(C)$ is a cell and the box $\pi(B)$ around $\pi(p)$ makes $\pi(C)$ open. By induction hypothesis, we know that $\pi(C)$ has type $(1)_{i=1,\dots,m-1}$. Since the fibre of $\pi(p) \in \pi(C)$ contains an open interval $]a_m, b_m[$, C has to have type $(1)_{i=1,\dots,m}$.

Let C now be an cell of type $(1)_{i=1,\dots,m}$. If $m = 1$, this means that C is an open interval. For larger m, we have two definable continuous functions f_1, f_2 such that $C = \{p = (p_1, \dots, p_m) \in M^m \colon f(p_1, \dots, p_{m-1}) < p_m < f(p_1, \dots, p_{m-1})\}$. The cell $\pi(C)$ has type $(1)_{i=1,\dots,m-1}$ and is open by induction assumption. For $p \in C$ let $B' = \prod_{i=0}^{m-1}]a_i, b_i[$ be an open neighborhood of $\pi(p) \in \pi(C)$. By restricting B', we may assume that there exists an $\varepsilon > 0$ such that $f_1(t) + \varepsilon < f_2(t) - \varepsilon$ for all $t \in B'$, since f_1, f_2 are continuous. Let $a_m = \sup\{f_1(t) \colon t \in B'\}$ and $b_m = \inf\{f_2(t) \colon t \in B'\}$. Since $f_1(t) - f_2(t) > 2\varepsilon$, $a_m - b_m \geq 2\varepsilon$, hence $a_m \neq b_m$. Now $B' \times]a_m, b_m[\subseteq C$ is an open neighborhood of p, hence C open.

(b) Show that if I_i is a non-empty open interval for $1 \leq i \leq m$, then $I_1 \times \cdots \times I_m$ is an open cell in M^m.

Solution:
We obtain $\prod I_i$ by taking f_1, f_2 as constant functions in every step, so $\prod I_i$ is a cell of type $(1, \dots, 1)$ and by (a) it is open.

Exercise 2

The goal of this exercise is to show that if $X \subset M^m$ is a definable set which is the union of finitely many cells C_i and if X has non empty interior, then one of the C_i is an open cell (without using cell decompositions!).

We do this by induction on m, so assume that the property holds for subsets of M^{m-1}. Assume that the interior of $X \subset M^m$ is not empty, and let $x_0 \in X$ be an interior point. Let $x_0' \in M^{m-1}$ be the projection on the first $m-1$ coordinates, and similarly use A' to denote the image of a subset $A \subset M^m$.

(a) Show that there exists an open neighborhood U of x_0' such that the fiber X_y of the projection $X \to M^{m-1}$ is infinite if $y \in U$.

Solution:
Since the interior X° is open, we can find an open box $B = \prod_{i=1}^m]a_i, b_i[\subseteq X^\circ$ containing x_0. We let $U = B'$, which now is an open neighborhood of x_0'. For every $y \in U$, we have $]a_m, b_m[\subseteq X_y^\circ \subseteq X_y = \{p \in X : p' = y\}$. This is an infinite set.

(b) Show that there exists $y \in U$ such that X_y is contained in the union of the C_i where C_i' is open in M^{m-1}.

Solution:
Note that if C_i is a cell such that C_i' is not open, then by exercise 1, C_i' is not of type $(1, \ldots, 1)$. This implies that C_i' has empty interior. By general topological facts, the union of all non-open C_i' also has emtpy interior. Thus

$$U \cap \bigcup_{C_i' \text{ not open}} C_i' \subseteq M^{m-1}$$

has empty interior, but U does not, so there exists

$$y \in U \setminus \bigcup_{C_i' \text{ not open}} C_i' = U \cap \bigcup_{C_i' \text{ open}} C_i'$$

Thus $X_y \subseteq U \cap \bigcup_{C_i' \text{ open}} C_i$.

(c) Conclude that some C_i is an open cell.

Solution:
We consider the finitely many cells C_i such that C_i' is open. Then $X_y \subseteq \bigcup_{C_i' \text{ open}} C_i$ by (b). Since X_y is infinite by (a), it there must be a C_i with open C_i' such that C_i contains infinitely many elements of X_y. This cell must therefore have type $c_m = 1$ and overall have type $(1)_{i=1,\ldots,m}$, as C_i' is of type $(1)_{i=1,\ldots,m-1}$. By exercise 1, C_i is open.

Exercise 3

We assume that $\mathcal{L}$ extends the language of ordered rings, so that M is an ordered ring. For any $n < m$, let $\pi_{m,n} \colon M^m \to M^n$ be the projection to the first n coordinates.

(a) For any cell $C \subset M^m$, show that there is a definable homeomorphism $C \to M^{\dim(C)}$.

Solution:
We use induction on m.

Since M is an ordered ring and o-minimal, M is a real closed field, (by the last exercise of Sheet 4). We first construct a definable homeomorphism $\varphi \colon]-1,1[\to M$ by

$$x \mapsto \frac{-x}{(x-1)(x+1)}.$$

If $M = \mathbb{R}$, then φ is a homeomorphism. Since being continuous and being bijective are definable properties, and since the theory of real closed fields is complete, φ is also a homeomorphism over other real closed fields.

Now we can modify this function to obtain homeomorphisms $\varphi_{a,b} \colon]a,b[\to M$ by $\varphi_{a,b}(t) = \varphi(\frac{t-a}{b-a})$. For open intervals that include infinity, we can define

$$\varphi_{-\infty,a}(t) = (t-a) - \frac{1}{t-a} = \varphi_{a,\infty}(t)$$

This shows the statement for $m = 1$, since the only cells are either points (which are definably homeomorphic to M^0) and open intervals.

Now assume that the statement holds for $m-1$, i.e. for every $\pi(C)$ there is a definable homeomorphism $\varphi_{\pi(C)} \colon \pi(C) \to M^{\dim(\pi(C))}$.

If $C \subseteq M^m$ has type $c_m = 0$, then it is definably homeomorphic to $\pi(C)$, which is a cell in M^{m-1} and hence definably homeomorphic to $M^{\dim(\pi(C))} = M^{\dim(C)}$. Composing two definable homeomorphisms results in another definable homeomorphism, hence C is definably homeomorphic to $M^{\dim(C)}$.

If $c_m = 1$, then there are continusous functions $f_1, f_2 \colon \pi(C) \to M$ such that $f_1 < f_2$ and $C = \{(x,y) \in M^{m-1} \times M \colon f_1(x) < y < f_2(x)\}$. We define

$$\begin{aligned} \varphi_C \colon \quad & C \to M^{\dim(\pi(C))} \times M = M^{\dim(C)} \\ & (x,y) \mapsto (\varphi_{\pi(C)}(x), \varphi_{f_1(x),f_2(x)}(y)) \end{aligned}$$

and see that φ_C is definable. Over $\mathbb{R}$ we can see that this function is also a homeomorphism, hence it is a homeomorphism over the real closed field M.

(b) Let $X \subset M^m$ be definable. Show that there is a definable map $\sigma\colon \pi_{m,n}(X) \to X$ such that $\pi_{m,n} \circ \sigma$ is the identity.

Solution:

Choose a cell decomposition $\mathcal{D}$ of X. Then $\mathcal{D}_n = \pi_{m,n}$ is also a cell decomposition and over every cell $C_n \in \mathcal{D}_n$ there are finitely many cells $C \in \mathcal{D}$. For every cell $C_n \in \mathcal{D}_n$ we pick the "lowest" cell C above C_n which for every c_i is defined by either $\Gamma(f)$ if $c_i = 0$ or $]f_1, f_2[$ if $c_i = 1$ in each step.

We define

$$\sigma\colon \pi_{m,n}(X) \to X$$
$$(x_1, \dots, x_n) \mapsto (x_1, \dots, x_n, \psi_{n+1}(x_1, \dots, x_n), \dots, \psi_m(x_1, \dots, \psi_{m-1}(x_1, \dots, x_{m-1}))$$

where $\psi_k(x_1, \dots, x_{k-1}) = f(x_1, \dots, x_{k-1})$ if $c_k = 0$ (and f is the defining function in this step), or

$$\psi_k(x_1, \dots, x_{k-1}) = \frac{f_1(x_1, \dots, x_{k-1}) + f_2(x_1, \dots, x_{k-1})}{2}$$

if $c_k = 1$ (and f_1, f_2 are defining functions in this step), or

$$\psi_k(x_1, \dots, x_{k-1}) = f_1(x_1, \dots, x_{k-1}) + 1$$

if $c_k = 1$ (and $f_2 = \infty$), or

$$\psi_k(x_1, \dots, x_{k-1}) = f_2(x_1, \dots, x_{k-1}) - 1$$

if $c_k = 1$ (and $f_1 = -\infty$), or

$$\psi_k(x_1, \dots, x_{k-1}) = 1$$

if $c_k = 1$ (and $f_1 = -\infty$ and $f_2 = \infty$). By construction, σ is well defined and definable. For every $x = (x_1, \dots, x_n)$ we have

$$\pi_{m,n}(\sigma(x)) = x.$$

Exercise 4

For a cell C, we define $\chi(C) = (-1)^{\dim(C)}$. For a finite family $\mathcal{C} = (C_i)_{i \in I}$ of disjoint cells in M^m, we define

$$\chi(\mathcal{C}) = \sum_{i \in I} \chi(C_i) = \sum_{k=0}^{m} (-1)^k n_k$$

where n_k is the number of cells of dimension k in $\mathcal{C}$.

(a) Let $\mathcal{D}$ be a cellular decomposition of a cell $C \subset M^m$. Show that

$$\chi(\mathcal{D}) = \chi(C).$$

(Hint: use induction on m, and sum over the projections of the cells in M^{m-1}, according to their type.)

Solution:
If $m = 1$, and $C \subseteq M$ is a cell, then it is either a point, in which case there is only one decomposition, or C is an open interval. If $C =]a, b[$, every cellular decomposition of C has to be of the form

$$\mathcal{D} = \{]a, t_1[\, \{t_1\},]t_1, t_2[\, , \{t_2\}, \dots ,]t_n, b[\, \}$$

by o-minimality. We have

$$\chi(\mathcal{D}) = n_0 - n_1 = -1 = (-1)^{\dim(C)} = \chi(C).$$

Now assume that $\chi(\mathcal{D}) = \chi(C)$ for every cell in M^{m-1}. For every cellular decomposition $\mathcal{D}$ of a cell $C \subseteq M^m$, we get a cellular decomposition $\pi(\mathcal{D})$ of the cell $\pi(C) \subseteq M^{m-1}$.

If the type of C ends with $c_m = 0$, then we have $\dim(C) = \dim(\pi(C))$ and hence $\chi(C) = \chi(\pi(C))$. For all $D \in \mathcal{D}$ we then also have $\chi(D) = \chi(\pi(D))$ for the same reason. In total we use the induction hypothesis to show

$$\chi(C) = \chi(\pi(C)) = \chi(\pi(\mathcal{D})) = \sum_{\pi(D) \in \pi(\mathcal{D})} \chi(\pi(D)) = \sum_{D \in \mathcal{D}} \chi(D) = \chi(\mathcal{D}).$$

If the type of C ends with $c_m = 1$, then $\dim(C) = \dim(\pi(C)) + 1$ and hence $\chi(C) = -\chi(\pi(C))$. For all $\tilde{D} \in \pi(\mathcal{D})$ we have

$$\{C_i \in \mathcal{D} \colon \pi(C_i) = \tilde{D}\} = \{]f_0, f_1[, \Gamma(f_1),]f_1, f_2[\, , \dots , \Gamma(f_n),]f_n, f_{n+1}[\, \} ,$$

where we denote $\Gamma(f)$ by a cell of type $(\dots, 0)$ defined by f and $]f_i, f_{i+1}[$ by a cell of type $(\dots, 1)$ defined by f_i, f_{i+1} for $f_i < f_{i+1}$, all of them with base D. Note that this set contains n many cells of dimension $\dim(\tilde{D})$ and $n + 1$ many cells of dimension $\dim(\tilde{D}) + 1$. The Euler-characteristic is then $\chi(\Gamma(f_i)) = \chi(\tilde{D})$ and $\chi(]f_i, f_{i+1}[\,) = -\chi(\tilde{D})$. We then have

$$\begin{aligned}
\chi(\mathcal{D}) &= \sum_{\tilde{D} \in \pi(\mathcal{D})} \sum_{\substack{C_i \in \mathcal{D} \\ \pi(C_i) = \tilde{D}}} \chi(C_i) = \sum_{\tilde{D} \in \pi(\mathcal{D})} \left(\sum_{i=1}^{n} \chi(\Gamma(f_i)) + \sum_{i=0}^{n} \chi(]f_i, f_{i+1}[\,) \right) \\
&= \sum_{\tilde{D} \in \pi(\mathcal{D})} n \cdot \chi(\tilde{D}) - (n+1) \cdot \chi(\tilde{D}) = \sum_{\tilde{D} \in \pi(\mathcal{D})} -\chi(\tilde{D}) \\
&= -\chi(\pi(\mathcal{D})) = -\chi(\pi(C)) = -(-1)^{\dim(\pi(C))} = (-1)^{\dim(C)} = \chi(C).
\end{aligned}$$

(b) Let $X \subset M^m$ be definable. Show that $\chi(\mathcal{D})$ is independent of $\mathcal{D}$ for all cellular decompositions of X. This common value is denoted $\chi(X)$.

Solution:
Given two cellular decompositions $\mathcal{D}_1$ and $\mathcal{D}_2$, we use the cellular decomposition theorem to obtain a new cellular decomposition $\mathcal{D}$ which is adapted to all the finitely many definable subsets of M^m that are elements in $\mathcal{D}_1$ and $\mathcal{D}_2$. We now use

$$\begin{aligned}\chi(\mathcal{D}_1) &= \sum_{C\in\mathcal{D}_1} \chi(C) = \sum_{C\in\mathcal{D}_1} \sum_{\substack{D\in\mathcal{D}\\ D\subseteq C}} \chi(D) = \sum_{D\in\mathcal{D}} \chi(D) \\ &= \sum_{C\in\mathcal{D}_2} \sum_{\substack{D\in\mathcal{D}\\ D\subseteq C}} \chi(D) = \sum_{C\in\mathcal{D}_2} \chi(C) = \chi(\mathcal{D}_2).\end{aligned}$$

(c) Show that if X_1 and X_2 are definable subsets of M^m, then

$$\chi(X_1 \cup X_2) = \chi(X_1) + \chi(X_2) - \chi(X_1 \cap X_2).$$

(Hint: first treat the case where X_1 and X_2 are disjoint.)

Solution:
If X_1, X_2 are disjoint, then we can choose any cellular decomposition $\mathcal{D}$ adapted to X_1 and X_2 to get $\chi(X_1 \cup X_2) = \chi(X_1) + \chi(X_2)$. Now in the general case, we apply the previous formula the following three disjoint unions

$$\begin{aligned}X_1 \cup X_2 &= X_1 \setminus X_2 \,\dot\cup\, X_2 \\ &= X_2 \setminus X_2 \,\dot\cup\, X_1 \\ &= X_1 \setminus X_2 \,\dot\cup\, X_2 \setminus X_1 \,\dot\cup\, X_1 \cap X_2\end{aligned}$$

to obtain

$$\begin{aligned}\chi(X_1 \cup X_2) &= \chi(X_1 \setminus X_2 \,\dot\cup\, X_2) + \chi(X_2 \setminus X_1 \,\dot\cup\, X_1) - \chi(X_1 \setminus X_2 \,\dot\cup\, X_2 \setminus X_1 \,\dot\cup\, X_1 \cap X_2) \\ &= \chi(X_2) + \chi(X_1) - \chi(X_1 \cap X_2).\end{aligned}$$

(d) Let $X \subset M^m$ be definable and let $n < m$. Show that for any $k \in \mathbb{Z}$, the set

$$\{a \in M^n \mid \chi(X_a) = k\}$$

is definable.

Solution:
Let $\mathcal{D}$ be a cellular decomposition of the definable set X, hence

$\pi_{m,n}(\mathcal{D})$ is a cellular decomposition of $\pi_{m,n}(X) \subseteq M^n$. Let $\tilde{C} \in \pi_{m,n}(\mathcal{D})$. For every $a \in \tilde{C}$, the number and types of cells in X_a is the same. Hence the function $\tilde{C} \to \mathbb{Z}, a \mapsto \chi(X_a)$ is constant. Let now

$$\mathcal{D}_k = \left\{\tilde{C} \in \pi_{m,n}(\mathcal{D})\colon \forall a \in \tilde{C}\colon \chi(X_a) = k\right\} \subseteq M^n.$$

which is a finite set of definable cells $\tilde{C}_i \in \pi_{m,n}(\mathcal{D})$. Let $\varphi_i(\underline{x})$ be the corresponding definable formulas, such that $\varphi_i(M^n) = \tilde{C}_i$. Then

$$\{a \in M^n\colon \chi(X_a) = k\} = \left(\bigvee_i \varphi_i(\underline{x})\right)(M^n).$$

is a definable set.