Solutions to Sheet 6

Let $\mathcal{L}$ be a language containing $\leq$ and let M be an o-minimal $\mathcal{L}$-structure. We say "definable" for "definable with parameters". This sheet is about point-set topology of definable sets.

Exercise 1

The goal of this exercise is to prove that definable sets in $\mathbb{R}^m$ are connected if and only if they are d-connected. Suppose that $M = \mathbb{R}$ with the usual interpretation of the order.

(a) Let $X \subseteq \mathbb{R}^m$ be a definable set which is connected *in the usual topological sense*. Show that X is d-connected.

Solution:
Let $U, V \subseteq X$ be two definable open sets with $U \cup V = X$ and $U \cap V = \emptyset$. Since X is connected, $U = \emptyset$ or $V = \emptyset$, hence X is d-connected.

(b) Let $C \subseteq \mathbb{R}^m$ be a cell. Show that C is connected in the usual topological sense.

Solution:
We do induction. For $m = 0$, points are connected. For $m = 1$, cells are open intervals (or points), which are connected in the usual topological sense. Now assume that the projection $C' \subseteq M^{m-1}$ is connected. Let U, V be two open sets with $U \cup V = C$ and $U \cap V = \emptyset$. For $y \in C'$ consider the fibre $C_y \subseteq C$ and note that we have the disjoint decomposition

$$(U \cap C_y) \cup (V \cap C_y) = C_y \qquad (U \cap C_y) \cap (V \cap C_y) = \emptyset$$

into open sets $(U \cap C_y)$ and $(V \cap C_y)$. The set C_y is either a point or an open interval, hence connected and hence $(U \cap C_y) = \emptyset$ or $(V \cap C_y) = \emptyset$. From this we get that the projections of U and V don't intersect $U' \cap V' = \emptyset$. Since also $U' \cup V' = C'$ and U' and V' are open, we use the induction hypothesis to conclude $U' = \emptyset$ or $V' = \emptyset$, which implies $U = \emptyset$ or $V = \emptyset$ proving that C is connected in the usual topological sense.

(c) Let $X \subseteq \mathbb{R}^m$ be a d-connected definable set and $\mathcal{D}$ a cellular decomposition of X. Assume that $X = U \cup V$ where U and V are disjoint open sets in X (for the usual topology, so $U = U_1 \cap X$ where $U_1 \subset \mathbb{R}^m$ is open, etc). Show that for any cell C of $\mathcal{D}$, we have $C \subset U$ or $C \subset V$.

Solution:
We have a disjoint decomposition $(U \cap C) \cup (V \cap C) = C$ into open sets $(U \cap C)$ and $(V \cap C)$. Since cells are connected (by (b)), we have either $(U \cap C) = \emptyset$, in which case $C \subseteq V$ or $(V \cap C) = \emptyset$ in which case $C \subseteq U$.

(d) Deduce that U and V are adapted to $\mathcal{D}$ and conclude that d-connected definable subsets $X \subseteq \mathbb{R}^m$ are connected in the usual sense.

Solution:
By (c), U and V are adapted to $\mathcal{D}$. Let U, V be open sets form a disjoint cover $X = U \cup V$ of the definable set X. Since U and V are adapted, U and V are definable. If X is d-connected, this means that $U = \emptyset$ or $V = \emptyset$, which means that X is also connected in the usual sense.

(e) Find an o-minimal structure M and a d-connected definable subset which is not connected in the usual topology.

Solution:
We can for example take $\mathcal{L} = \{<\}$ and the $\mathcal{L}$-structure $M = \mathbb{R} \setminus \{0\}$ with the usual interpretation of $<$. We note that this is o-minimal, since the order is a dense linear order without endpoints and definable sets are finite unions of points and intervals.

We now have that M is d-connected, but not connected, because $]-\infty, 0[$ and $]0, \infty[$ are open disjoint subsets whose union is M. Note that these two open subsets are not d-connected.

Exercise 2

We assume that $\mathcal{L}$ extends the language of ordered rings, so that we know that M is a real closed (ordered) field. Let $m \geq 1$ be an integer. We denote $|x| = \max(x, -x)$ for $x \in M$, and we put $\|x\| = \max(|x_i|)$ for $x = (x_i) \in M^m$. A subset $X \subset M^m$ is *bounded* if and only if there exists $A \in M$ such that $\|x\| \leq A$ for all $x \in A$.

(a) Prove that the topology of M^m is generated by the sets of the form $\{x \in M^m \mid \|x - x_0\| < \delta\}$ for $x_0 \in M^m$ and $\delta > 0$ in M.

Solution:
The usual topology on M^m is generated by products of intervals. The balls

$$B(x_0, \delta) = \{x \in M^m \colon \|x - x_0\| < \delta\}$$

are of the form $B(x_0, \delta) = \prod]x_0 - \delta, x_0 + \delta[$ and hence open sets. On the other hand, if $x \in \prod_i]a_i, b_i[$, then we can choose

$$\delta_x = \min\{|x - a_i|, |x - b_i| : i = 1, \ldots, m\},$$

then

$$\bigcup_{x \in X} B(x, \delta_x) = \prod_i]a_i, b_i[$$

which shows that the open balls generate the topology of M^m.

(b) Let $X \subset M^m$ be definable and let x_0 be an element of M^m belonging to the closure of X. This means that there is a sequence $x_n \in X$ with $\lim_{n\to\infty} x_n = x_0$. Sequences are not definable, but their role can be replaced by definable maps $\gamma\colon]0, c[\to X$ with $\lim_{x\to 0} \gamma(x) = x_0$.

(i) Assume that $x_0 \in \overline{X} \setminus X$. Prove that there exists a non-empty open interval $I =]0, c[$ and a definable map $\gamma\colon I \to X$ such that $\|x_0 - \gamma(t)\| = t$ for all $t \in I$.
Hint: use Exercise 3 of Exercise Sheet 5.

Solution:
Let $x_0 \in \overline{X} \setminus X$. The set

$$D_{x_0} = \{\|x_0 - x\| \in M : x \in X\}$$

is a definable subset of M and hence by o-minimality a finite union of points and open intervals. Since $x_0 \in \overline{X} \setminus X$, D_{x_0} contains infinitely many arbitrarily small positive values. If $b = \sup\{x \in M : \forall y : 0 < y \leq b, y \in D_{x_0}\}$, then we have that the open interval $]0, b[\subseteq D_{x_0}$. Now the set

$$S = \{(t, x) \in]0, b[\times X : \|x_0 - x\| = t\} \subseteq M \times M^m$$

is also definable and we have the projection

$$\pi_{m+1,1} : S \to M.$$

By exercise 3 of sheet 5, there is a definable map $\sigma\colon]0, b[= \pi_{m+1,1}(S) \to S$ that satisfies $\pi_{m+1,1} \circ \sigma(t) = t$ for all $t \in]0, b[$, which means that $\sigma(t) = (t, \gamma(t))$ for some $\gamma(t) \in X$. The map $\gamma\colon]0, b[\to X$ is definable. For all $t \in]0, b[$ we have $\sigma(t) = (t, \gamma(t)) \in S$, hence $\|x_0 - \gamma(t)\| = t$.

(ii) Prove that there exists $c > 0$ in M and a continuous definable map $\gamma\colon]0, c[\to X$ such that $\lim_{t\to 0} \gamma(t) = x_0$.
Hint: consider separately the case when $x_0 \in X$.

Solution:
When $x_0 \in X$, we can define $\gamma(t) = x_0$ which satisfies $\lim_{t\to 0}\gamma(t) = x_0$. If $x_0 \in \overline{X} \setminus X$, we use (i) to obtain a definable map $\gamma\colon]0, b[\to X$ with $\|x_0 - \gamma(t)\| = t$. For $\lim_{t\to 0}\gamma(t) = x_0$, we have to show that for all open neighborhoods U of x_0, there is a $T \in]0, b[$ such that for all $t \in]0, T[$ we have $\gamma(t) \in U$. By part (a) it is enough to consider $U = B(x_0, \delta)$. We can choose $T = \delta$ and we get that for all $t \in]0, \delta[$, $\|x_0 - \gamma(t)\| = t < \delta$, hence $\gamma(t) \in B(x_0, \delta)$.

To get a definable map, we may use the monotonoicity theorem to restrict γ to $]0, c[\subseteq]0, b[$ such that γ is continuous.

(c) Let $C \subset M^m$ be a *bounded* cell and $\overline{C}$ its closure. Let $\pi\colon M^m \to M^{m-1}$ be the projection that omits the last coordinate. Show that $\overline{\pi(C)} = \pi(\overline{C})$.

Hint: deal separately with the cases where C is a graph or the space between two graphs of continuous definable functions on $\pi(C)$; apply the previous exercise to show that if $a \in M^{m-1}$ is in the closure of $\pi(C)$, then it is in $\pi(\overline{C})$.

Solution:
For any continuous map, such as π, we have $\pi(\overline{C}) \subseteq \overline{\pi(C)}$. Now let $x_0 \in \overline{\pi(C)} \setminus \pi(C)$. We have to show that $x_0 \in \pi(\overline{C})$. By part (b) we get a definable continuous map $\gamma\colon]0, c[\to \pi(C)$ with $\lim_{t\to 0}\gamma(t) = x_0$. Now we upgrade it to $\gamma'\colon]0, c[\to C$ by defining

$$\gamma'(t) = (t, f(t))$$

if $C = \Gamma(f)$ and

$$\gamma'(t) = \left(t, \frac{f(t) + g(t)}{2}\right)$$

if $C =]f, g[$. Note that we assumed C to be bounded, and hence f and g are bounded. Thus $\lim_{t\to 0}\gamma'(t)$ exists in $\overline{C}$. Then $\pi(\lim_{t\to 0}\gamma'(t)) = \lim_{t\to 0}\gamma(t) = x_0 \in \pi(\overline{C})$.

(d) Let $X \subset M^m$ be closed, bounded and definable. Let $f\colon X \to M^n$ be definable and continuous. The goal of this exercise is to prove that $f(X)$ is bounded.

(i) Assuming that $f(X)$ is not bounded, show that there exists a definable map $g\colon M \to X$ such that $\|f(g(t))\| > t$ for all $t \in M$.

Solution:
We consider the definable sets

$$T = \{t \in]1, \infty[\colon \exists x \in X\colon \|f(x)\| = t\}$$
$$S = \{(t, x) \in T \times X\colon \|f(x)\| = t\} \subseteq M \times M^m.$$

and note that since $f(X)$ is not bounded, T contains arbitrarily large elements, and hence by o-minimality there is an $a \in M$ ($a > 1$) such that $]a, \infty[\subseteq T$. We now use Exercise 3 of Sheet 5, to obtain a definable map $\sigma\colon]a, \infty[\to S$ such that for all $t \in]a, \infty[$ we have $\pi_{m+1,1}(\sigma(t)) = t$. We now define a map $g\colon M \to X$:
If $t > a - 1$, let $g(t) \in X$ such that $\sigma(t+1) = (t+1, g(t)) \in S$. In this case

$$\|f(g(t))\| = t + 1 > t.$$

If $t \leq a - 1$, then we set $g(t) \in X$ such that $\sigma(a) = (a, g(t)) \in S$. Here also

$$\|f(g(t))\| = a \geq t + 1 > t.$$

(ii) Show that the limit of $g(t)$ as $t \to +\infty$ in $\overline{X}$ exists. *Hint: apply the monotonicity theorem to each coordinate of g.*

Solution:
Assume g is given in coordinates as $g\colon M \to X \subseteq M^m$ as $t \mapsto (g_1(t), \ldots, g_m(t))$. By the monotonicity theorem there is a decomposition of M into finitely many points and open intervals such that g_i restricted to these open intervals is strictly monotonically increasing, decreasing or constant. Let $b_i \in M$ be sucht that $]b_i, \infty[$ is the rightmost interval in the decomposition of M corresponding to g_i. If g_i is strictly monotonically increasig, let $g_i(\infty) = \sup\{g_i(t)\colon t \in]b_i, \infty[\}$, if g_i is strictly monotonically decreasig, let $g_i(\infty) = \inf\{g_i(t)\colon t \in]b_i, \infty[\}$, if g_i is constant, let $g_i(\infty) = g_i(b_i + 1)$. Since X is assumed to be bounded, we have $g_i(\infty) \in M$ also bounded. We now have $\lim_{t\to\infty} g(t) = (g_1(\infty), \ldots, g_m(\infty)) \in \overline{X}$.

(iii) Deduce a contradiction, hence the result.

Solution:

As f and the norm are continuous, we have for all $t \in M$ that $\lim_{t\to\infty} \|f(g(t))\| = \|f(\lim_{t\to\infty} g(t))\| \in M$, which is a finite element and contradicts $\|f(g(t))\| > t$ from part (i).