O-minimality and diophantine applications

ETH Zürich – Autumn semester 2019

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These are preliminary lecture notes for the course "O-minimality and diophantine applications" given at ETH Zürich in autumn semester 2019. They will be updated regularly. Comments and remarks are very welcomed.

Contents

1 Outline of the course

The overall goal of this course is to provide an introduction to o-minimality and to prove results needed for diophantine applications. The first part of the course will be devoted to the definition of o-minimal structures and to proving the cell decomposition theorem, which is crucial for describing the shape of subsets of an o-minimal structure. In the second part of the course, we will prove the Pila-Wilkie counting theorem. The last part will be devoted to diophantine applications, with the proof by Pila and Zannier of the Manin-Mumford conjecture and, if time permits, a sketch of the proof by Pila of the André-Oort conjecture for products of modular curves.

1.1 Main references

We will mainly follow Scanlon's paper.
[Sca17] Thomas Scanlon. O-minimality as an approach to the André-Oort conjecture. In *Around the Zilber-Pink Conjecture/Autour de La Conjecture de Zilber-Pink*, volume 52 of *Panor. Synthèses*, pages 111–165. Soc. Math. France, Paris, 2017

For o-minimal structures and the cell decomposition theorem, the main reference is van den Dries' book.
[vdD98] Lou van den Dries. *Tame Topology and o-Minimal Structures.* Cambridge University Press, Cambridge, 1998.

About the Pila-Wilkie counting theorem, the Pila-Zannier strategy, and its application by Pila to the product of modular curves, here are the original papers.
[PW06] Jonathan Pila and Alex J. Wilkie. The rational points of a definable set. *Duke Math. J.*, 133(3):591–616, 2006

[PZ08] Jonathan Pila and Umberto Zannier. Rational points in periodic analytic sets and the Manin-Mumford conjecture. *Atti Accad. Naz. Lincei Rend. Lincei Mat. Appl.*, 19(2):149–162, 2008

[Pil11] Jonathan Pila. O-minimality and the André-Oort conjecture for $\mathbb{C}^n$. *Ann. of Math. (2)*, 173(3):1779–1840, 2011

One can also consult the following book.
[JW15] Gareth O. Jones and Alex J. Wilkie, editors. *O-Minimality and Diophantine Geometry*, volume 421 of *London Mathematical Society Lecture Note Series*. Cambridge University Press, Cambridge, 2015

1.2 O-minimal structures

Definition 1.1. A *semi-algebraic* subset of $\mathbb{R}^n$ is a finite boolean combination of sets of the form $\{x \in \mathbb{R}^n \mid f(x) > 0\}$, where $f \in \mathbb{R}[X_1, \ldots, X_n]$ is a polynomial with real coefficients. In particular, it is a finite union of sets of the form

$$\{x \in \mathbb{R}^n \mid f(x) = 0, g_1(x) > 0, \ldots, g_s(x) > 0\},$$

with $f, g_1, \ldots, g_s \in \mathbb{R}[X_1, \ldots, X_n]$.

O-minimal geometry aims at mimicking properties of semi-algebraic sets in a more general context. For this introduction we stick to the field of real numbers, but we will adopt later a more general definition.

Definition 1.2. An *o-minimal structure* on the field $\mathbb{R}$ is a family $\mathcal{S} = (\mathcal{S}_n)_{n\in\mathbb{N}}$ where each $\mathcal{S}_n$ is a collection of subsets of $\mathbb{R}^n$ such that for each $n \geq 0$,

(1) $\mathcal{S}_n$ is a boolean algebra, *i.e.* $\mathcal{S}_n$ is non-empty and if $A, B \in \mathcal{S}_n$, then $A \cup B \in \mathcal{S}_n$ and $\mathbb{R}^n \backslash A \in \mathcal{S}_n$,

(2) if $A \in \mathcal{S}_n$, then $A \times \mathbb{R}$ and $\mathbb{R} \times A$ belong to $\mathcal{S}_{n+1}$,

(3) $\{(x_1, \ldots, x_n) \in \mathbb{R}^n \mid x_1 = x_n\} \in \mathcal{S}_n$,

(4) if $A \in \mathcal{S}_{n+1}$, and $\pi : \mathbb{R}^{n+1} \to \mathbb{R}^n$ is the projection map to the first n-th coordinates, then $\pi(A) \in \mathcal{S}_n$,

(5) $\mathcal{S}_n$ contains the semi-algebraic subsets of $\mathbb{R}^n$,

(6) $\mathcal{S}_1$ consists of *finite* unions of points and open intervals.

We say that $A \in \mathcal{S}_n$ is definable (in $\mathcal{S}$). We say that a map $f : A \subset \mathbb{R}^n \to \mathbb{R}^{n'}$ is definable if its graph $\Gamma(f) \subset \mathbb{R}^{n+n'}$ is definable.

Remark 1.3. (1) Collection of sets satisfying $(1) - (4)$ are called *structures* (on $\mathbb{R}$) and are the topic of model theory.

(2) The key property to check that $\mathcal{S}$ is a structure is (4). We'll see that it is related to quantifier elimination.

(3) If $\mathcal{S}$ is a structure on $\mathbb{R}$ such that the graph of the addition, multiplication and of the order relation are definable, together with every element of $\mathbb{R}$, then $\mathcal{S}$ contains every semi-algebraic set.

(4) The term "o-minimal" comes from "order-minimal" in the sense that the definable sets in $\mathcal{S}_1$ are built using the order relation of $\mathbb{R}$ only.

(5) If we are given any collection of sets $\mathcal{S}$ (sometimes called a pre-structure), then we can close it under conditions (1) – (4) in order to form the smallest structure $\tilde{\mathcal{S}}$ containing $\mathcal{S}$. Even if $\mathcal{S}_1$ satisfies (6), in general $\tilde{\mathcal{S}_1}$ does not !

Examples 1.4. (1) (Tarski-Seidenberg) The collection of all semi-algebraic sets is stable under projections and in particular is an o-minimal structure.

(2) (Denef-van den Dries) Let $\mathbb{R}_{\text{an}}$ be the smallest structure containing semi-algebraic sets and, for every $U \subset \mathbb{R}^n$ an open set containing the box $[0,1]^n$, and every analytic function $f : U \to \mathbb{R}$, that contain the graph of f restricted to $[0,1]^n$. Then $\mathbb{R}_{\text{an}}$ is an o-minimal structure.

(3) (Wilkie) Let $\mathbb{R}_{\text{exp}}$ be the smallest structure containing semi-algebraic sets and the graph of the full exponential function. Then $\mathbb{R}_{\text{exp}}$ is an o-minimal structure.

(4) (van den Dries-Miller) Let $\mathbb{R}_{\text{an,exp}}$ be the smallest structure containing $\mathbb{R}_{\text{an}}$ and $\mathbb{R}_{\text{exp}}$. Then $\mathbb{R}_{\text{an,exp}}$ is o-minimal. Applications of o-minimality to diophantine geometry mainly use this structure.

(5) (non-example) Let $\mathcal{S}$ be the smallest structure containing the semi-algebraic sets and $\mathbb{Z}$. Then $\mathcal{S}$ contains all the so-called projective sets, in particular all Borel sets. It is not at all o-minimal. In particular, a structure containing the (full) graph of the since function is not o-minimal.

1.3 Cell decomposition

We fix for the rest of this section an o-minimal structure $\mathcal{S}$ over $\mathbb{R}$, and "definable" means definable in $\mathcal{S}$.

Proposition 1.5. *(exercise)*

(1) If $X \subset \mathbb{R}^n$ is definable, then its topological closure $\bar{X}$ is definable.

(2) If $f : \mathbb{R} \to \mathbb{R}$ is definable, then the limits $\lim_{x\to b^-} f(x)$ and $\lim_{x\to b^+} f(x)$ exist (in $\mathbb{R} \cup \{-\infty, +\infty\}$) for every $b \in \mathbb{R}$.

The following result is the most important result about o-minimal structures. A *cell* is defined inductively as follows.

(1) The cells of $\mathbb{R}$ are singletons and open intervals.

(2) If $C \subset \mathbb{R}^n$ is a cell, and $f, g : C \to \mathbb{R}$ are definable continuous maps with $f < g$, then the followings sets are cells : the graph of f, $\{(c,x) \in C \times \mathbb{R} \mid f(c) < x < g(c)\}$, $\{(c,x) \in C \times \mathbb{R} \mid f(c) < x\}$, $\{(c,x) \in C \times \mathbb{R} \mid x < f(c)\}$.

Theorem 1.6 (Cell decomposition)**.** *If $X \subset \mathbb{R}^n$ is definable, then there is a cell decomposition of X,* i.e. *there is a finite partition of $\mathbb{R}^n$ into cells inducing a partition of X. If $f : X \to \mathbb{R}$ is a definable map, the decomposition can be chosen such that f is continuous on each of the cells contained in X.*

The map f can in fact be required to be of class $\mathcal{C}^k$ on each cell, for any fixed $k \in \mathbb{N}$.

1.4 Pila-Wilkie counting theorem

Applications of o-minimality to diophantine geometry mostly comes from the following theorem, due to Pila and Wilkie.

For $x \in \mathbb{Q}$, say $x = \frac{p}{q}$, $p, q \in \mathbb{Z}$ in lowest terms, the (naive) *height* of x is $H(x) := \max\{|p|, |q|\}$. For $x = (x_1, \dots, x_n) \in \mathbb{Q}^n$, the height of x is $H(x) := \max\{H(x_1), \dots, H(x_n)\}$.

For any $X \subset \mathbb{R}^n$, define $X(\mathbb{Q}, H) := \{x \in X \cap \mathbb{Q}^n \mid H(x) \leq H\}$ and observe that it is a finite set.

For any $X \subset \mathbb{R}^n$, define X^{alg}, the algebraic part of X to be the union of infinite, connected semi-algebraic subsets of X. Note that since it may be an infinite union, X^{alg} is in general not definable, even if X is definable. It may happen that $X^{\mathrm{alg}} = X$ (e.g. if X is itself a connected semi-algebraic set) or $X^{\mathrm{alg}} = \emptyset$ (e.g. if X is a transcendental curve). Define the transcendental part of X to be $X^{\mathrm{tran}} := X \backslash X^{\mathrm{alg}}$.

The Pila-Wilkie counting theorem asserts that there are very few points of bounded height in the transcendental part of a definable set.

Theorem 1.7 (Pila-Wilkie counting theorem)**.** *Let $X \subset \mathbb{R}^n$ be a set definable in an o-minimal structure on $\mathbb{R}$. Then for every $\varepsilon > 0$, there is a constant $C = C(X, \varepsilon)$ such that for every $H \geq 1$,*

$$\# X^{\mathrm{tran}}(\mathbb{Q}, H) \leq C H^{\varepsilon}.$$

Remark 1.8.

(1) We will in fact prove a stronger, uniform version, for definable families of sets.

(2) The term H^{ε} cannot be improved in general, but it is conjectured that for some specific o-minimal theories (e.g. $\mathbb{R}_{\exp}$), it can be replaced by $\log(H)^{\alpha}$ for some $\alpha \geq 0$.

1.5 Manin-Mumford conjecture

Recall that an abelian variety A is a proper group variety. Fix an abelian variety A defined over a subfield of $\mathbb{C}$. The group of torsion points, $A^{\mathrm{tor}} \subset A(\mathbb{C})$, is always Zariski dense in A. The Manin-Mumford conjecture, first proved by Raynaud, is the following statement.

Theorem 1.9. *Let A be an abelian variety defined over $\mathbb{C}$. Let $X \subset A$ be an irreducible subvariety. If $X(\mathbb{C}) \cap A^{\mathrm{tor}}$ is Zariski-dense in X, then X is a translate of an abelian subvariety of A.*

There are many related statements/conjectures, *e.g.* for copies of the multiplicative group, for moduli space of zbelian varieties, or more generally for Schimura varieties. In that case the conjecture is called the André-Oort conjecture.

In 2008, Pila and Zannier found a new proof of the Manin-Mumford conjecture, relying on the Pila-Wilkie counting theorem, with the advantage of providing a strategy to prove these more general conjectures. This method was successfully applied to prove the Andre-Oort conjecture for products of modular curves by Pila, for moduli spaces of abelian varieties by Pila-Tsimerman, other cases by Masser-Zannier, Habegger-Pila, Klingler-Ullmo-Yafaev . . .

Here is an outline of the strategy of proof of the Manin-Mumford conjecture, assuming that A is defined over a number field K. Fix a subvariety X of A, and assume that X does not contain any translate of a (non-trivial) abelian subvariety of A. We need to show that $X(\mathbb{C}) \cap A_{\mathrm{tor}}$ is finite. Since A is an abelian variety, say of dimension g, there is a $\mathbb{Z}$-lattice $\Lambda \subseteq \mathbb{C}^g$ such that $\mathbb{C}^g/\Lambda$ is complex-analytically isomorphic to $A(\mathbb{C})$. Via this identification, the projection map $\pi : \mathbb{C}^g \to A(\mathbb{C})$ is then complex analytic. Viewing $\mathbb{C}$ as $\mathbb{R}^2$, one can choose a compact fundamental domain $F$of $\mathbb{C}^g$ under the action of Λ, such that the restriction of π to F is definable in $\mathbb{R}_{\mathrm{an}}$. Set $W = \pi_{|F}^{-1}(X(\mathbb{C}))$. Under our hypothesis on X, the Ax-Lindemann-Weierstrass theorem asserts that the algebraic part of W is empty, hence $W = W^{\mathrm{tran}}$. The torsion points on A correspond via π to points in $\mathbb{Q}\Lambda$, which can be identified, up to a linear transformation, to rational points in W. One then needs to show that W contains only finitely many rational points. By the Pila-Wilkie theorem, for every $\varepsilon > 0$ there is a constant C such that for any $H \geq 1$,

$$\#W(\mathbb{Q}, H) \leq CH^{\varepsilon}.$$

On the other hand, if $x \in W$ is a rational point of height exactly H, then $\pi(x)$ is a torsion point of A of order H. Recall that A is defined over a number field K. A theorem of Masser asserts that there exist constants $c > 0$ and $\rho > 0$ such that for any torsion point P of A of order H, we have

$$[K(P) : K] \geq cH^{\rho}.$$

For any such P, all the Galois-conjugates of P are also torsion points, providing at least cH^{ρ} different rational points in W. Comparing with the bound provided by the Pila-Wilkie theorem (say for $\varepsilon = \rho/2$), we see that all the rational points in W are of height bounded by some H_0, hence there are only finitely many of them.

2 Introduction to model theory

We introduce here basic notions of model theory that will be needed later.

2.1 Language and structures

A model-theoretic, or first-order structure, consists of a formal language $\mathcal{L}$, together with an interpretation of this language.

Definition 2.1. A *language* $\mathcal{L}$ is the data of a set $\mathcal{C}$ of *constants symbols*, a set $\mathcal{R}$ of *relation symbols*, a set $\mathcal{F}$ of *function symbols*, and maps $n_{\mathcal{R}} : \mathcal{R} \to \mathbb{N}^*$ and $n_{\mathcal{F}} : \mathcal{F} \to \mathbb{N}^*$, indicating the arity of each relation and function symbol.

Remark 2.2. Such a data is sometimes called the *signature* of the language $\mathcal{L}$.

Definition 2.3. An $\mathcal{L}$-structure $\mathfrak{M}$ is the data of a *non-empty* set M, and for each constant symbol $c \in \mathcal{C}$, the data of an element $c^{\mathfrak{M}} \in M$, for each n-ary relation symbol $R \in \mathcal{R}$, an n-ary relation $R^{\mathfrak{M}}$ on M (*i.e.* a subset $R^{\mathfrak{M}} \subset M^n$), and for each function symbol $f \in \mathcal{F}$ or arity n, a data of a function $f^{\mathfrak{M}} : M^n \to M$.

Remark 2.4. The distinction between syntax - formal language, symbols - and semantic - their interpretation - is the starting point of first-order logic. Even if we are ultimately more interested in structures with "standard" interpretation of the symbols, it is in the possibility of considering "non-standard" structures that the strength of model theory arise.

Example 2.5. (1) Define $\mathcal{L}_{\text{gp}}$ to be the language $\mathcal{L} := 1, \cdot, ()^{-1}$ with 1 a constant symbol, $\cdot$ a binary function symbol, and $()^{-1}$ a unary function symbol. Any group can be viewed as an $\mathcal{L}_{\text{gp}}$-structure, by interpreting the symbols respectively as the unit element, the multiplication law and the inverse map.

(2) Define $\mathcal{L}_{\text{ring}}$ to be the language $\mathcal{L}_{\text{ring}} = \{0, 1, +, -, \cdot\}$, with 0,1 constant symbols, $+,\cdot$ binary function symbols and $-$ an unary function symbols. Any ring can be viewed as an $\mathcal{L}_{\text{ring}}$-structure.

(3) The language of ordered rings $\mathcal{L}_{\text{ring},<}$ is $\mathcal{L}_{\text{ring}}$ with an additional binary relation symbol "$<$".

We can view the set $\mathbb{R}$ as an $\mathcal{L}_{\text{ring},<}$-structure by interpreting the constants 0 and 1 by "themselves", the relation symbol "$<$" by the usual order relation on $\mathbb{R}$, and the function symbols "$+$" and "$\cdot$" by the usual addition and multiplication maps on $\mathbb{R}$. This structure will be denoted by $\mathbb{R}_{\text{sa}}$.

Definition 2.6. A *term with variables* $x_1, \ldots, x_n$ in the language $\mathcal{L}$ (or $\mathcal{L}$-term) is defined inductively with the following rules :

- $x_1, \ldots, x_n$ and every constant symbols $c \in \mathcal{C}$ are terms;
- if $t_1, \ldots, t_m$ are terms with variables $x_1, \ldots, x_n$ and $f \in \mathcal{F}$ is a function symbol with arity m, then $f(t_1, \ldots, t_m)$ is a term with variables $x_1, \ldots, x_n$.

Definition 2.7. A *formula with free variables* $x_1, \ldots, x_n$ in the language $\mathcal{L}$ (or $\mathcal{L}$-formula) is defined inductively with the following rules :

(1) if $t_1, \ldots, t_m$ are terms with variables $x_1, \ldots, x_n$ and $R \in \mathcal{R}$ is a relation of arity m, then "$t_1 = t_2$" and "$R(t_1, \ldots, t_m)$" are $\mathcal{L}$-formulas with free variables $x_1, \ldots, x_n$;

(2) if φ and ψ are $\mathcal{L}$-formula with free variables $x_1, \ldots, x_n$, then "$\varphi \wedge \psi$", "$\varphi \vee \psi$", "$\neg\varphi$", "$\varphi \to \psi$" are $\mathcal{L}$-formulas with free variables $x_1, \ldots, x_n$;

(3) if φ is a $\mathcal{L}$-formula with free variables $x_1, \ldots, x_n$, then "$\exists x_1 \varphi$" and "$\forall x_1 \varphi$" are $\mathcal{L}$-formulas with free variables $x_2, \ldots, x_n$.

Remark 2.8. (1) The connectives $\wedge, \vee$, and $\neg$ are respectively called conjunction "and", disjunction "or", negation.

(2) There is some redundancy in our set of connectives, since every formula could be expressed using only $\wedge$, $\neg$ and $\exists$. We can also add a connective for the implication "$\to$".

(3) Formulas defined using only rule (1) are called *atomic formulas*, those defined using only rules (1) and (2) are called *quantifier-free formulas*.

(4) We are only allowing to quantify over variables (and *not* over formulas). This is what makes our logic *first-order*.

(5) A formula without free variables is called a *sentence*.

(6) If φ is a formula with free variables $x_1, \ldots, x_n$, or t a term with variables $x_1, \ldots, x_n$, we usually write $\varphi(x_1, \ldots, x_n)$ and $t(x_1, \ldots, x_n)$.

(7) It is important to specify the set of variables of a term, and the set of free variables of a formula, since one can always add "dummy" variables.

Definition 2.9 (Interpretation)**.** Given an $\mathcal{L}$-structure $\mathfrak{M} = (M, \ldots)$, we define the *interpretation* of terms and formulas as follows. An $\mathcal{L}$-term $t(x_1, \ldots, x_n)$ is interpreted as a map $t^{\mathfrak{M}} : M^n \to M$ defined inductively as follows :

(1) constant symbols $c \in \mathcal{C}$ are interpreted as the constant map equal to $c^{\mathfrak{M}}$, a variable x_i is interpreted as the map $(x_1, \ldots, x_n) \in M^n \mapsto x_i$;

(2) if $t_1, \ldots, t_m$ are terms with already defined interpretation, and $f \in \mathcal{F}$ is a symbol of m-ary function, then the interpretation of $f(t_1, \ldots, t_m)$ is the map

$$(x_1, \ldots, x_m) \in M^n \mapsto f^{\mathfrak{M}}(t_1^{\mathfrak{M}}(x_1, \ldots, x_n), \ldots, t_m^{\mathfrak{M}}(x_1, \ldots, x_n)).$$

An $\mathcal{L}$-formula φ with free variables $x_1, \ldots, x_n$ is interpreted as a subset $\varphi(\mathfrak{M}) \subset M^n$ defined inductively as follows :

(1) the formulas "$t_1 = t_2$" and "$R(t_1, \ldots, t_m)$" are interpreted respectively as

$$\left\{(x_1, \ldots, x_n) \in M^n \mid t_1^{\mathfrak{M}}(x_1, \ldots, x_n) = t_2^{\mathfrak{M}}(x_1, \ldots, x_n)\right\}$$

and

$$\left\{(x_1, \ldots, x_n) \in M^n \mid R^{\mathfrak{M}}(t_1^{\mathfrak{M}}(x_1, \ldots, x_n), \ldots, t_m^{\mathfrak{M}}(x_1, \ldots, x_n))\right\};$$

(2) if φ and ψ are $\mathcal{L}$-formulas with free variables $x_1, \ldots, x_n$ such that $\varphi(\mathfrak{M}), \psi(\mathfrak{M}) \subset M^n$ are already defined, then the interpretations of "$\varphi \wedge \psi$", "$\varphi \vee \psi$" and "$\neg\varphi$" are respectively

$$\varphi(\mathfrak{M}) \cap \psi(\mathfrak{M}), \varphi(\mathfrak{M}) \cup \psi(\mathfrak{M}) \text{ and } M^n \backslash \varphi(\mathfrak{M});$$

(3) if φ is a $\mathcal{L}$-formula with free variables $x_1, \ldots, x_n$ such that $\varphi(\mathfrak{M}) \subset M^n$ is already defined, then the interpretation of "$\exists x_1 \varphi$" is

$$\left\{(x_2, \ldots, x_n) \in M^{n-1} \mid \text{ there exists } x_1 \in M \text{ such that } (x_1, x_2, \ldots, x_n) \in \varphi(\mathfrak{M})\right\}$$

and the interpretation of "$\forall x_1 \varphi$" is

$$\left\{(x_2, \ldots, x_n) \in M^{n-1} \mid \text{ for every } x_1 \in M, (x_1, x_2, \ldots, x_n) \in \varphi(\mathfrak{M})\right\}.$$

Remark 2.10. (1) Two different formulas can have the same interpretation.

(2) Considering M^0 as a singleton, the definition also makes sense for sentences (formulas without free-variables). If φ is an $\mathcal{L}$-sentence such that $\varphi(\mathfrak{M})$ is non-empty, we say that φ is *true* in $\mathfrak{M}$, or that $\mathfrak{M}$ models φ, denoted by $\mathfrak{M} \models \varphi$. Otherwise, we say that φ is *false* in $\mathfrak{M}$.

(3) If $\varphi(x)$ is an $\mathcal{L}$-formula and a a tuple of elements of M we write $\mathfrak{M} \models \varphi(a)$ if $a \in \varphi(\mathfrak{M})$. Accordingly, if $\varphi(x, y)$ is an $\mathcal{L}$-formula, and b a tuple of elements of M, we define $\varphi(\mathfrak{M}, b) := \{a \in M^n \mid \mathfrak{M} \models \varphi(a, b)\}$. Observe that this is compatible with the previous definition, if we consider M as an $\mathcal{L}(M)$-structure, where $\mathcal{L}(M)$ is the language $\mathcal{L}$ extended by adding a new constant symbol for each element of M.

(4) A subset $X \subset M^n$ is said to be *$\mathcal{L}$-definable without parameters* if there is an $\mathcal{L}$-formula $\varphi(x_1, \ldots, x_n)$ such that $X = \varphi(\mathfrak{M})$. We call a X *basic definable* (resp. *quantifier-free definable*) if it is definable by an atomic (resp. a quantifier-free) formula.

(5) A subset $X \subset M^n$ is said to be *$\mathcal{L}$-definable with parameters* if it is of the form $\varphi(\mathfrak{M}, b)$, where $\varphi(x, y)$ is an $\mathcal{L}$-formula and b a tuple of elements of M. Equivalently, X is definable without parameters in M viewed as an $\mathcal{L}(M)$-structure. By "definable", we usually mean "definable with parameters".

Example 2.11. (1) Terms in the language of groups $\mathcal{L}_{\text{gp}}$ "are" words $x_{i_1}^{\pm 1} \cdot x_{i_2}^{\pm 1} \cdots x_{i_s}^{\pm 1}$.

(2) Terms in the language of rings $\mathcal{L}_{\text{ring}}$ "are" polynomials with coefficients in $\mathbb{Z}$. In any ring viewed as an $\mathcal{L}_{\text{ring}}$-structure, they are interpreted as the associated polynomial maps. (Note that it is cheating since we are not supposed to remove unnecessary parenthesis in a term. This won't pose any problem as long as one only considers $\mathcal{L}_{\text{ring}}$-structures that are actual rings.)

For the $\mathcal{L}_{\text{ring}}$-formula $\varphi(x, y, z) :=$ "$x^2 + y^2 = 0 \wedge z^2 = 2$", we have $\varphi(\mathbb{Q}) = \emptyset$, $\varphi(\mathbb{R}) = \{(0, 0, \pm\sqrt{2})\}$, $\varphi(\mathbb{C}) = \{(x, \pm ix, \pm\sqrt{2}) \mid x \in \mathbb{C}\}$.

(3) The $\mathcal{L}_{\text{ring}}$-formula $\varphi(x, y) :=$ "$\exists z z \neq 0 \wedge y = x + z^2$" defines the graph of the order relation in $\mathbb{R}$, but $\mathbb{C}^2$ in $\mathbb{C}$.

(4) As described in Example 2.5, the set $\mathbb{R}$ can be viewed as a structure $\mathbb{R}_{\text{sa}}$ in the language of ordered rings $\mathcal{L}_{\text{ring},<} = \mathcal{L}_{ring} \cup \{<\}$. The quantifier-free definable sets (with parameters) are exactly the semi-algebraic sets. We will see later that *every* definable set in this structure is semi-algebraic.

(5) The $\mathcal{L}_{\text{ring}}$-formula $\varphi(x) :=$ "$\exists y x = y^2$" defines the set of squares. In $\mathbb{R}$, it correspond to the set of non-negative elements, while in $\mathbb{C}$ is is the whole $\mathbb{C}$.

Definition 2.12. (1) An *$\mathcal{L}$-theory $\mathcal{T}$* is a set of $\mathcal{L}$-sentences. If $\mathfrak{M}$ is an $\mathcal{L}$-structure, we say that $\mathfrak{M}$ is a model of $\mathcal{T}$ if for every $\varphi \in \mathcal{T}$, $\mathfrak{M} \models \varphi$. We denote it by $\mathfrak{M} \models \mathcal{T}$.

(2) A theory is said to be *satisfiable* if it admits a model. It is said to be *finitely satisfiable* if every finite subset of it admits a model.

(3) We say that an $\mathcal{L}$-sentence φ is true in $\mathcal{T}$, written $\mathcal{T} \models \varphi$, if φ is true in every model of $\mathcal{T}$.

Cultural remark 2.13. At the syntaxic level, one can define the notion of *formal proof* of φ in $\mathcal{T}$ which is a *finite* sequence of $\mathcal{L}$-sentences, ending by φ, such that each sentence of the sequence is either in $\mathcal{T}$, either deduced from the previous ones by "basic deduction rules" (*e.g.* from φ and $\varphi \to \psi$, one can deduce ψ, or from $\forall x\varphi(x)$ one can deduce $\exists x\varphi(x)$). We write $\mathcal{T} \vdash \varphi$ if φ admits a formal proof in $\mathcal{T}$. A direct unraveling of definitions shows that $\mathcal{T} \vdash \varphi$ implies that $\mathcal{T} \models \varphi$. However, it is a non-trivial theorem, called *Gödel's completness theorem* that the converse is also true : $\mathcal{T} \models \varphi$ implies that $\mathcal{T} \vdash \varphi$. It can be used to deduce the compactness theorem of the next section.

Example 2.14. (1) In the language $\mathcal{L}_{\mathrm{gp}}$, we can consider the theory of groups, consisting of the formula

$$\forall x, y, z(x \cdot y) \cdot z = x \cdot (y \cdot z) \wedge \forall x(x \cdot x^{-1} = x^{-1} \cdot x = 1 \wedge x \cdot 1 = 1 \cdot x = 1)$$

Any group is a model of this theory.

(2) In the language $\mathcal{L}_{ring}$, we can consider the theory of fields (using a finite number of formulas). If we add for each $n > 0$, the formula "$\forall a_0, \dots, a_{n-1} \exists x \sum_{i=0}^{n-1} a_i x^i + x^n = 0$", one obtain the theory of algebraically closed fields ACF. Its models are algebraically closed fields. One can also specify the characteristic of the field (for characteristic zero, one needs an infinite number of sentences).

(3) In the language $\mathcal{L}_{<} = \{<\}$, we can consider the theory DLO of dense linear orders without endpoints, stating that "$<$" is a total order relation, without endpoints, and "$\forall x, y(x < y \to \exists z(x < z < y))$". The sets $\mathbb{Q}$ and $\mathbb{R}$, with their usual order relation are models of DLO.

(4) In the language $\mathcal{L}_{\mathrm{ring},<} = \mathcal{L}_{\mathrm{ring}} \cup \{<\}$ of ordered rings, we can consider the theory of ordered fields, consisting, in addition to the field axioms and the fact that $<$ is a total order relation, the formulas $\forall x, y, z(x < y \to x + z < y + z)$ and $\forall x, y, z(0 < x < y \wedge 0 < z \to xz < yz)$.

We can extend it to the theory RCF of real closed fields by stating additionally that the field $R[X]/(X^2 + 1)$ is algebraically closed (check that it can be be stated by first-order formulas), or equivalently by stating that every polynomial of odd degree admits a root.

(5) If $\mathfrak{M} = (M, \dots)$ is an $\mathcal{L}$-structure, we can consider its theory $\mathrm{Th}(\mathfrak{M})$, defined as the set of $\mathcal{L}$-sentences that are true in $\mathfrak{M}$.

2.2 Compactness theorem

The one and only theorem of abstract model theory that we will need is the following theorem.

Theorem 2.15 (Compactness)**.** *Let $\mathcal{T}$ be an $\mathcal{L}$-theory. If $\mathcal{T}$ is finitely satisfiable, then $\mathcal{T}$ is satisfiable.*

Note that the converse is obviously true.

Cultural remark 2.16. The term *compactness* refers to the fact that the space of *types* is compact. We won't need this, but an n-type in an $\mathcal{L}$-theory $\mathcal{T}$ is a set of $\mathcal{L}$-formulas with n free variables that is finitely satisfiable (in a model of $\mathcal{T}$). A complete type is a type p such that for every formula $\varphi(x)$, either $\varphi(x) \in p$ or $\neg\varphi(x) \in p$. The set of complete n-types is endowed with a topology whose basic open sets are sets of types containing a given formula. Then the compactness theorem says that this topological space is compact.

Example 2.17. Consider the theory $\mathcal{T}$ of $\mathbb{R}$ in the language $\mathcal{L}_{\text{sa}} = (\mathbb{R}, +, -, \cdot, <)$. Consider the language $\mathcal{L}' = \mathcal{L}_{\text{sa}} \cup \{\varepsilon\}$, with ε a constant symbol. Define the $\mathcal{L}'$-theory $\mathcal{T}'$ to be

$$\mathcal{T} \cup \{0 < \varepsilon\} \cup \{\varepsilon < 1/n\}_{n \in \mathbb{N}^*}.$$

Then if $\mathcal{T}_0 \subset \mathcal{T}'$ is a finite subset of $\mathcal{T}'$, it contains only finitely many sentence of the form $\varepsilon < 1/n$. Hence if n_0 is the biggest $n \in \mathbb{N}^*$ such that the sentence $\varepsilon < 1/n$ belongs to $\mathcal{T}_0$, we can view $\mathbb{R}$ as an $\mathcal{L}'$-structure, by interpreting the constant symbol ε as $\frac{1}{n_0+1}$. Then such a structure is a model of $\mathcal{T}_0$. By the compactness theorem, $\mathcal{T}'$ admits a model, say $\mathfrak{M} = (R, \dots)$. Such an R is an ordered field extension of $\mathbb{R}$, containing an infinitesimal element ε_0 (the interpretation of ε) that is strictly positive but smaller than every real number. By definition, the field R satisfies every $\mathcal{L}_{\text{sa}}$-sentence satisfied in $\mathbb{R}$, and is usually called (a) field of non-standard real numbers. With the help of ε_0, one can do non-standard analysis in R, and every $\mathcal{L}_{\text{sa}}$-sentence proven that way will be true in $\mathbb{R}$. Note that it also shows that the property of being Archimedean is not a first order property of $\mathbb{R}$.

We will also use the compactness theorem in situations similar to the following example, where out of a finiteness assumption, we get uniformity for "free".

Example 2.18. Let $\mathcal{T}$ be an $\mathcal{L}$-theory, and $\varphi(x, y)$ be an $\mathcal{L}$-formula. Suppose that for every model $\mathfrak{M} = (M, \dots)$ of $\mathcal{T}$, and every $b \in M$, the set $\varphi(\mathfrak{M}, b)$ is finite. Then there is a constant $k \in \mathbb{N}$ such that for every model $\mathfrak{M} = (M, \dots)$ of $\mathcal{T}$, and every $b \in M$, the set $\varphi(\mathfrak{M}, b)$ has at most k elements.

Indeed, suppose for the sake of contradiction that this is not true. Consider the language $\mathcal{L}' = \mathcal{L} \cup \{c_i\}_{i \in \mathbb{N}} \cup \{t\}$, where c_i and t are constants symbols. Consider the following $\mathcal{L}'$-theory $\mathcal{T}'$:

$$\mathcal{T}' = \mathcal{T} \cup \{c_i \neq c_j\}_{i \neq j} \cup \{\varphi(c_i, t)\}.$$

Let $\mathcal{T}_0$ be a finite subset of $\mathcal{T}'$. Since $\mathcal{T}_0$ is finite, there are only finitely many c_i that appears in $\mathcal{T}_0$, say there are N_0 of them. By hypothesis, there is a model $\mathfrak{M}_0$ of $\mathcal{T}$, and $b_0 \in M_0$, such that $\varphi(\mathfrak{M}_0, b_0)$ has more that N_0 elements. View M_0 as an $\mathcal{L}'$-structure by interpreting t by b_0, the N_0 different c_i that appears in $\mathcal{T}_0$ as distinct elements of $\varphi(\mathfrak{M}_0, b_0)$, and the other c_i as your favorite element of M_0. Then M_0 is a model of $\mathcal{T}_0$. By the compactness theorem, there is a model $\mathfrak{M}'$ of $\mathcal{T}'$. Such a model is also a model of $\mathcal{T}$. By definition, if t_0 is the interpretation of t in $\mathfrak{M}'$, the set $\varphi(\mathfrak{M}', t_0)$ is infinite, since it contains the interpretation of all the c_i, which are pairwise distinct.

Exercise 2.19. Adapt the previous argument to show that a satisfiable theory that admits an infinite model admits models of arbitrary large cardinality.

To prove the compactness theorem, we need the following notion of filter and ultrafilter, that you might have encountered in an analysis class.

Definition 2.20. A *filter* $\mathcal{F}$ on a set I is a non-empty set of subsets of I satisfying the following axioms :

(1) $\mathcal{F}$ is non-empty, $\emptyset \notin \mathcal{F}$

(2) if $A, B \in \mathcal{F}$, then $A \cap B \in \mathcal{F}$;

(3) if $A \subset B \subset I$ and $A \in \mathcal{F}$, then $B \in \mathcal{F}$.

An *ultrafilter* $\mathcal{F}$ on a set I is a filter on I such that for every subset $A \subset I$, either A or $I \backslash A$ is in $\mathcal{F}$.

A *non-principal ultrafilter* on a set I is an ultrafilter that does not contain any set of the form $\{a\}$, for $a \in I$. In other terms, it is an ultrafilter that contains the Fréchet filter, consisting of every cofinite subset (complement of a finite subset) of I.

Remark 2.21. Any set satisfying (1) and (2) can be extended in a unique way such that it is a filter. Moreover, the axiom of choice implies that any filter is contained in an ultrafilter.

Example 2.22. The set of all neighborhoods of a point in a topological space X is a filter on X.

Let $\mathcal{L}$ be a language, $\{\mathfrak{M}_i\}_{i\in I}$ a collection of $\mathcal{L}$-structures, where I is an infinite set, and $\mathcal{U}$ a non-principal ultrafilter on I. We define the ultraproduct of the $\{\mathfrak{M}_i\}_{i\in I}$ with respect to $\mathcal{U}$, denoted by $\prod_{i\in I} \mathfrak{M}_i/\mathcal{U}$, as follows. Consider the binary relation $\sim$ on the product $\prod_{i\in I} M_i$ defined by

$$(a_i)_{i\in I} \sim (b_i)_{i\in I} \iff \{i \in I \mid a_i = b_i\} \in \mathcal{U}.$$

One can check (exercise) that $\sim$ is an equivalence relation, and we define $\prod_{i\in I} M_i/\mathcal{U}$ as the quotient of $\prod_{i\in I} M_i$ by $\sim$. If $a_i \in M_i$ for $i \in I$, we denote by $(a_i)_{i\in I}/\sim \in \prod_{i\in I} M_i/\mathcal{U}$ the class of $(a_i)_{i\in I}$ modulo $\sim$.

We endow $\prod_{i\in I} M_i/\mathcal{U}$ with a $\mathcal{L}$-structure as follows. If c is a constant symbol, interpreted as $c_i \in M_i$, then we interpret c as $(c_i)_{i\in I}/\sim$ in $\prod_{i\in I} M_i/\mathcal{U}$.

If R is a symbol of an n-ary relation, we interpret it as

$$(a_i)_{i\in I}/\sim \in R \iff \left\{i \in I \mid a_i \in R^{\mathfrak{M}_i}\right\} \in \mathcal{U}$$

(exercise: check that it is well defined).

If f is a symbol of an n-ary function, we interpret it as

$$f((a_i)_{i\in I}/\sim) := (f^{\mathfrak{M}_i}(a_i)_{i\in I})/\sim$$

(exercise : check that it is well defined).

Theorem 2.23 (Łoś). *Consider the ultraproduct $\prod_{i \in I} \mathfrak{M}_i/\mathcal{U}$ as above. Then for any $\mathcal{L}$-formula $\varphi(x)$,*

$$\prod_{i \in I} \mathfrak{M}_i/\mathcal{U} \models \varphi((a_i)_{i \in I}/\sim) \iff \{i \in I \mid \mathfrak{M}_i \models \varphi(a_i)\} \in \mathcal{U}.$$

Proof. Here is a sketch of the proof, with details left as an exercise. One shows first (by induction) that the terms are interpreted as one expects (*i.e.* that their interpretation satisfy a formula similar to the one for functions). One then proceeds to the proof of the theorem, by induction as in the definition of formulas. One first shows the result for atomic formulas, which follows from the definitions and the "correct" interpretation of terms. One then proceeds to the induction. Assume that the result holds for $\varphi(x)$ and $\psi(x)$, with x a tuple of variables. One has the result for $(\varphi \wedge \psi)(x)$ since the intersection of two sets in a filter is in the filter. For the negation, we have

$$\prod_{i \in I} \mathfrak{M}_i/\mathcal{U} \models \neg\varphi((a_i)_{i \in I}/\sim \iff \{i \in I \mid \mathfrak{M}_i \models \varphi(a_i)\} \notin \mathcal{U} \iff \{i \in I \mid \mathfrak{M}_i \models \neg\varphi(a_i)\} \in \mathcal{U},.$$

where we used for the last equivalence the maximality property of ultrafilters.

Suppose the result is known for $\psi(x, y)$ (where x is a single variable). We then have

$$\begin{aligned} \prod_{i \in I} \mathfrak{M}_i/\mathcal{U} \models \exists x \varphi(x, (b_i)_{i \in I}/\sim) \quad &\iff \quad \text{there exists } (a_i)_{i \in I} \text{ such that } \{i \in I \mid \mathfrak{M}_i \models \varphi(a_i, b_i)\} \in \mathcal{U} \\ &\iff \quad \{i \in I \mid \mathfrak{M}_i \models \exists x \varphi(x, b_i)\} \in \mathcal{U}, \end{aligned}$$

where for the last implication, we use the inclusion property of filters, and for the reverse implication we use the axiom of choice. Since the other logical connectors can be expressed in terms of $\wedge, \neg$ and $\exists$, the theorem is proven. □

We can now proceed to the proof of the compactness theorem.

Proof of Theorem 2.15. Let $\mathcal{T}$ be an $\mathcal{L}$-theory that is finitely satisfiable. We want to construct a model of $\mathcal{T}$. If $\mathcal{T}$ is finite, there is nothing to prove, so assume that $\mathcal{T}$ is infinite. Let I be a set indexing the finite subsets $\mathcal{T}_i$ of $\mathcal{T}$. Choose for each $i \in I$ a model $\mathfrak{M}_i$ of $\mathcal{T}_i$. Set $A_i := \{j \in I \mid \mathcal{T}_i \subset \mathcal{T}_j\}$. The collection of all A_i satisfies the conditions (1) and (2) of filters, so by Remark 2.21, we can choose an ultrafilter $\mathcal{U}$ on I containing all A_i. Note that $\mathcal{U}$ is non-principal, otherwise it would contain a set $\{i\}$, which would contradict the finiteness of $\mathcal{T}_i$. Consider the ultraproduct $\prod_{i \in I} \mathfrak{M}_i/\mathcal{U}$. Let us show that this $\mathcal{L}$-structure is a model of $\mathcal{T}$. If φ is an $\mathcal{L}$-sentence, by Łoś' theorem, we have that

$$\prod_{i \in I} \mathfrak{M}_i/\mathcal{U} \models \varphi \iff \{i \in I \mid \mathfrak{M}_i \models \varphi\} \in \mathcal{U}.$$

But if $\varphi \in \mathcal{T}$, then $\{\varphi\}$ is a finite subset of $\mathcal{T}$, hence there is an i such that $\{\varphi\} = \mathcal{T}_i$. Since $A_i \subset \{j \in I \mid \mathfrak{M}_j \models \varphi\}$, by property (3) of filters, $\{j \in I \mid \mathfrak{M}_j \models \varphi\} \in \mathcal{U}$, hence $\prod_{i \in I} \mathfrak{M}_i/\mathcal{U} \models \varphi$, which concludes the proof. □

2.3 Quantifier elimination

A set definable using quantifiers is usually much more complicated than one definable without quantifier. If one wants a good description/understanding of all definable sets in a given theory, the theory needs to admit quantifier elimination.

Definition 2.24. (1) We say that an $\mathcal{L}$-theory $\mathcal{T}$ admits *quantifier-elimination* in $\mathcal{L}$ if for every $\mathcal{L}$-formula $\varphi(x)$ (where x is a tuple of variables), there is a quantifier-free $\mathcal{L}$-formula $\psi(x)$ such that $\mathcal{T} \models \forall x(\varphi(x) \leftrightarrow \psi(x))$.

(2) We say that an $\mathcal{L}$-structure $\mathfrak{M}$ admits quantifier elimination if $\mathrm{Th}(\mathfrak{M})$ admits quantifier elimination.

More concretely, if means that any definable set can be defined by a quantifier-free formula.

Remark 2.25. (1) We can always obtain quantifier elimination for "free" by extending the language. Indeed, define $\mathcal{L}'$ to be the language $\mathcal{L}$ extended by a new relation symbol $R_{\varphi(x)}$ for each $\mathcal{L}$-formula $\varphi(x)$. Define an $\mathcal{L}'$ -theory $\mathcal{T}'$ to be $\mathcal{T}$ together with "$\forall y(R_{\varphi(x)}(y) \leftrightarrow \varphi(y))$" for each $\mathcal{L}$-formula $\varphi(x)$. Every $\mathfrak{M}$, model of $\mathcal{T}$, can be canonically extended as a model $\mathfrak{M}'$ of $\mathcal{T}'$, by interpreting $R_{\varphi(x)}$ as $\varphi(\mathfrak{M})$. Note that sets definable in $\mathfrak{M}$ and $\mathfrak{M}'$ are the same. The theory $\mathcal{T}'$ admits quantifier elimination.

(2) In view of the previous remark, if one start with a given theory $\mathcal{T}$ in a language $\mathcal{L}$, one tries to find the "smallest" (or "most natural") language $\mathcal{L}'$ extending $\mathcal{L}$, such that $\mathcal{T}$ extends as an $\mathcal{L}'$-theory without changing the definable sets and admits quantifier elimination in $\mathcal{L}'$.

Example 2.26. (1) Chevalley's theorem, stating that the projection of a constructible set in an algebraically closed field is again constructible, is equivalent to the fact that the theory ACF of algebraically closed fields in the language of rings $\mathcal{L}_{\mathrm{ring}} = \{0, 1, +, -, \cdot\}$ admits quantifier-elimination.

(2) The theory DLO of dense total orders without endpoints admits quantifier elimination in the language $\mathcal{L} = \{<, c\}$.

(3) Tarski-Seidenberg's theorem, stating that the projection of a semi-algebraic set in a real closed field is again semi-algebraic, is equivalent to the fact that the theory RCF of real closed fields admits quantifier elimination in the language $\mathcal{L}_{\mathrm{ring},<} = \{0, 1, +, -, \cdot, <\}$.

(4) Let $\mathcal{L}_{\mathrm{an}}$ be the language containing $\mathcal{L}_{\mathrm{ring},<}$ and for every $n \in \mathbb{N}^*$, an n-ary function symbol f for any analytic function $f : U \to \mathbb{R}$ defined on an open set U containing $[0, 1]^n$.

We view $\mathbb{R}$ as an $\mathcal{L}_{\mathrm{an}}$-structure by interpreting as usual the symbols in $\mathcal{L}_{\mathrm{ring},<}$, and by interpreting each function symbol "f" of an analytic functionby the map $x \mapsto f(x)$ for $x \in [0, 1]^n$, and $x \mapsto 0$ for $x \notin [0, 1]^n$. This structure is denoted by $\mathbb{R}_{\mathrm{an}}$.

A theorem of Denef and van den Dries [Dv88] asserts that $\mathbb{R}_{\text{an}}$ admits quantifier elimination in the language $\mathcal{L}_{\text{an},D}$, where D is a binary function symbol interpreted as $D(x,y) = x/y$ if $0 \leq x/y \leq 1$, 0 otherwise.

Exercise 2.27 (Sturm's theorem)**.** Let $p \in \mathbb{R}[X]$ be a (univariate) square free polynomial. Consider the following sequence : $p_0 := p$, $p_1 = p'$, $p_{i+2} :=$ the opposite of the remainder of the Euclidian division of p_i by p_{i+1}. For $a \in \mathbb{R}$, define $V(x)$ to be the number of sign changes in the sequence $p_0(a), p_1(a), \ldots$ (ignoring zeroes). If $a, b \in \mathbb{R}$, $a < b$ such that $p(a)p(b) \neq 0$, show that the number of roots of p in the interval $[a, b]$ is equal to $V(a) - V(b)$. (Hint : examine what happens when x passes through a root of one of the p_i.)

More generally, the same result holds if we replace $\mathbb{R}$ by a real closed field, and this result can be used to show quantifier-elimination for RCF. See for example [BCR98] for more details.

3 O-minimal structures

We introduce in this section the general notion of an o-minimal structure, present the main examples and derive some consequences of the definition.

3.1 Definition and examples

Definition 3.1. (1) An *o-minimal structure* is an $\mathcal{L}$-structure $\mathfrak{R} = (R, <, \ldots)$ in a language $\mathcal{L}$ containing "$<$" such that $(R, <)$ is a dense total order without endpoints and such that the definable subsets of R (with parameters) are exactly the quantifier-free definable sets (with parameters) in the language $\mathcal{L}_o = \{<\}$, that is, finite unions of points and intervals.

(2) An *o-minimal expansion of an ordered field* is an o-minimal structure $\mathfrak{R} = (R, <, \ldots)$ in a language $\mathcal{L} = \{0, 1, +, -, \cdot, <, \ldots\}$ extending the language of ordered fields such that R is an ordered field.

(3) An $\mathcal{L}$-theory $\mathcal{T}$ in a language $\mathcal{L} = \{<, \ldots\}$ is said to be *o-minimal* if every model of $\mathcal{T}$ is o-minimal.

Remark 3.2. (1) Note that if $\mathbb{R}$ is the underlying set of an o-minimal expansion of an ordered field, then its definable sets (with parameters) correspond exactly to the Definition 1.2 given in the introduction.

Since we are mostly interested in definable sets in o-minimal structures, from this point on we could forget most of the model-theoretic premises, and adopt a more naive definition as in the introduction. The use of logical formula is nevertheless handy when we want to manipulate definable sets.

(2) An $\mathcal{L}$-structure $\mathfrak{R}$ such that $\mathrm{Th}(\mathfrak{R})$ is o-minimal is sometimes called *strongly o-minimal.* As we will show later, a striking consequence of the definition is that *any* o-minimal structure is strongly o-minimal.

(3) There is no "maximal" o-minimal structure on $\mathbb{R}$. A theorem of Rolin, Speissegger and Wilkie [RSW03] asserts that there are functions $f, g : \mathbb{R} \to \mathbb{R}$ such that the structures $(\mathbb{R}, <, +, -, \cdot, f)$ and $(\mathbb{R}, <, +, -, \cdot, g)$ are both o-minimal, but not $(\mathbb{R}, <, +, -, \cdot, f, g)$.

Example 3.3. The $\mathcal{L}_<$-theory DLO of dense linear orders without endpoints is o-minimal, since its admits quantifier-elimination.

Example 3.4. The structure $\mathbb{R}_{\text{sa}}$ of semi-algebraic subsets of $\mathbb{R}$ is o-minimal. More generally, the theory RCF of real closed fields is o-minimal.

Proof. The stability of semi-algebraic sets under projections follows from the Tarski-Seidenberg theorem. Hence definable sets of $\mathbb{R}$ are finite boolean combinations of sets for the form

$$\{x \in \mathbb{R} \mid f(x) > 0\},$$

with $f \in \mathbb{R}[X]$. By continuity of polynomial maps on $\mathbb{R}$, they are finite union of points and intervals. □

Example 3.5. The structure $\mathbb{R}_{\text{an}}$ of Example 2.26 is o-minimal. This follows from quantifier-elimination and the following fact, proven by induction on terms :

if $t(x)$ is (the interpretation of) an $\mathcal{L}_{\text{an},D}$-term (with a single variable x), then there exists $\varepsilon > 0$ such that either $t(x) = 0$ on $(0, \varepsilon)$, or $t(x) = x^n f(x)$ on $(0, \varepsilon)$, with $n \in \mathbb{Z}$ and f an invertible power series converging on $(-\varepsilon, \varepsilon)$.

Example 3.6. Wilkie in [Wil96] shows that the structure $\mathbb{R}_{\exp} = (\mathbb{R}, +, -, \cdot, <, \exp)$ is o-minimal. Although does not admits quantifier-elimination in a "simple" language, the proof proceeds by showing that every definable set is of the form $\exists y_1, \ldots, y_n \varphi(y_1, \ldots, y_n, x)$.

Example 3.7. The structure $\mathbb{R}_{\text{an,exp}}$ of the real numbers in the language $\mathcal{L}_{\text{an}} \cup \{\exp\}$ is o-minimal, by a theorem of van den Dries and Miller [vM94]. So far, all the applications of o-minimality to diophantine geometry use the $\mathbb{R}_{\text{an,exp}}$ (or smaller) structure.

3.2 Topology

We fix for the rest of the section an o-minimal structure $\mathfrak{R} = (R, <, \ldots)$. We will use the notation $\bar{R} = R \cup \{-\infty, +\infty\}$, and extend the order relation $<$ to $\bar{R}$ in the usual way.

By an *interval* in R, we mean a non-empty open interval, *i.e.* a set of the form $]a, b[:= \{x \in R \mid a < x < b\}$, for $a, b \in \bar{R}, a < b$.

We consider the *order topology* on R, which is the topology with basis for the open sets given by the intervals. Similarly R^n is equipped with the product topology, a basis of which is given by open boxes $]a_1, b_1[\times \cdots \times]a_n, b_n[$.

Lemma 3.8. *(1) If $f : A \subset R^n \to R^m$ is a definable map, then A and $f(A)$ are definable.*

(2) If $A \subset R$ is definable, then $\sup(A), \inf(A)$ are well-defined (in $\bar{R}$). The boundary ∂A of A is a finite set of points. If $a_1 < \cdots < a_n$ are the points of the boundary of A, and $a_0 = -\infty$, $a_{n+1} = +\infty$, then for each $i = 0, \ldots, n$, the interval (a_i, a_{i+1}) is either included in A, or disjoint from A.

(3) *If $A \subset R^n$ is definable, then its topological closure* $\mathrm{cl}(A)$ *and its interior* $\mathrm{int}(A)$ *are definable.*

(4) *If $f : A \subset R^n \to R^m$ is a definable map and A is open, then the set of $a \in A$ such that f is continuous at a is definable.*

Remark 3.9. In the definition of o-minimality, one can replace the condition on definable subsets of R by the fact that every definable subset of R that is neither $\emptyset$ nor R has a boundary that is a nonempty finite set of points.

Proof. (1) If $\varphi(x, y)$ is a formula defining the graph of f, then A and $f(A)$ are respectively defined by $\exists y \varphi(x, y)$ and $\exists x \varphi(x, y)$.

(2) Direct consequence of the definition of o-minimality.

(3) The topological closure of A is defined by the following formula

$$\{(x_1, \ldots, x_n) \in R^n \mid \forall y_1, \ldots, y_n, z_1, \ldots, z_n (y_1 < x_1 < z_1, \ldots, y_n < x_n < z_n) \to$$
$$(\exists a \in A, \exists y_1 < a_1 < z_1, \ldots, y_n < a_n < z_n)\}$$

which is first-order if A is defined by a first-order formula. One can write similarly a first-order formula for the interior of A, or use the complement.

(4) If $\varphi(x, y)$ is a formula defining f, then the set of continuity points of f is the set of $a = (a_1, \ldots, a_n)$ satisfying the following formula :

$$\forall z_1, \ldots, z_m, z'_1, \ldots, z'_m, b_1, \ldots, b_m (\varphi(a, b) \wedge z_1 < b_1 < z'_1, \ldots, z_m < b_m < z'_m) \to$$
$$\exists x_1, \ldots, x_n, x'_1, \ldots, x'_n (x_1 < a_1 < x'_1, \ldots, x_n < a_n < x'_n)$$
$$\forall a'_1, \ldots, a'_n, b'_1, \ldots, b'_m (x_1 < a'_1 < x'_1, \ldots, x_n < a'_n < x'_n)$$
$$(\varphi(a', b') \to z_1 < b'_1 < z'_1, \ldots, z_m < b'_m < z'_m)).$$

□

Definition 3.10. A set $X \subset R^n$ is said to be *definably connected* if X is definable and X is not the disjoint union of two non-empty open definable subsets of X.

Remark 3.11. (1) If $R = \mathbb{R}$, then we will see later that for definable sets, definable connectedness coincide with the usual topological notion of connectedness.

(2) If R is an ultrapower of $\mathbb{R}$, then one can show that the set of infinitesimal elements ($x \in R$ such that $0 < x < r$ for every $r \in \mathbb{R}$) is an open and closed subset of R. In particular, R is *not* connected for the order topology. This justifies the introduction of definable connectedness.

Lemma 3.12. *(1) The definably connected subsets of R are the empty set, the intervals, the sets of the form $[a, b[$, $]a, b]$, $[a, b]$ for $a, b \in \bar{R}$, $a < b$.*

(2) The image of a definably connected set by a definable continuous map is definably connected.

Proof. Exercise. □

In particular, this lemma implies that the intermediate value theorem holds for definable continuous maps : the image of $[a, b]$ by a continous definable function contains every value between $f(a)$ and $f(b)$.

4 Cell decomposition theorem

We prove in this section the cell decomposition theorem, following closely Chapter 3 of [van98]. We fix for the whole section an o-minimal structure $\mathfrak{R} = (R, <, \dots)$.

4.1 Monotonicity theorem

The following result is an important step toward the cell decomposition theorem, due to Pillay and Steinhorn [PS84]. We call a map $f :]a, b[\to R$ *strictly monotone* if it is either constant, strictly increasing or strictly decreasing. We say that f is *locally strictly monotone* at $x \in]a, b[$ (resp. *locally constant*, ...) if there is a subinterval of $]a, b[$ containing x such that f is strictly monotone (resp. ...) on it. We say that f is *locally strictly monotone* (resp. ...) if it is locally strictly monotone (resp. ...) at every $x \in]a, b[$.

Theorem 4.1 (Monotonicity theorem). *Let $f : I = (a, b) \to R$ be a definable map. Then there exist elements $a_0 = a < a_1 < \cdots < a_s = b$ of R such that the restriction of f on each interval $]a_i, a_{i+1}[$ is continuous and strictly monotone.*

Corollary 4.2. *(1) For any $c \in [a, b]$, the limits $\lim_{x\to c^-} f(x)$ and $\lim_{x\to c^+} f(x)$ exist in $\bar{R}$.*

(2) Let $f : [a, b] \to R$ be a definable and continuous map (for $a, b \in R$) , then f reaches a maximum and minimum on $[a, b]$.

Proof. Exercise. □

We will derive the theorem from the following two lemmas, dealing with a definable map $f : I \to R$, where I is an interval.

Lemma 4.3. *There is a subinterval of I on which f is monotone.*

Lemma 4.4. *If f is strictly monotone, then f is continous on a subinterval of I.*

Assuming those lemmas, the proof of the monotonicity theorem goes as follows.

Proof of Theorem 4.1. Consider the set

$$A := \{x \in]a, b[\mid f \text{ is locally strictly monotone at } x\} .$$

Then $(a, b)\backslash A$ is finite. Indeed, otherwise it would contains an interval I. But by Lemmas 4.3 and 4.4, there would be a subinterval of I such that f is continous strictly monotone on it.

Since $]a, b[\backslash A$ is finite, we can split $]a, b[$ into finitely many intervals and suppose that $A =]a, b[$. Up to further splitting $]a, b[$, we can suppose that f is either locally consant on $]a, b[$, either locally strictly increasing on (a, b), either locally strictly decreasing on $]a, b[$.

In the first case, for every $x_0 \in]a, b[$, consider

$$s(x_0) := \sup \{x \mid x_0 < x < b, f \text{ is constant on } [x, b[\} .$$

Then $s(x_0) = b$, otherwise $s(x_0) < b$ would contradict the local constancy of f at $s(x_0)$. Hence f is constant on $[x_0, b[$. Similarly, f is constant on $]a, x_0]$, hence f is constant. The two other cases are similar, replacing "constant by "strictly increasing" and "stricly decreasing". $\square$

Proof of Lemma 4.4. Assume that f is strictly increasing. Then $f(I)$ is infinite, hence contains an interval $K \subset f(I)$. Choose two points $c, d \in K$, $c < d$ and consider their preimage by $f : c', d' \in I$ such that $f(c') = c, f(d') = d$. Then f is an order preserving bijection between $]c', d'[$ and $]c, d[$, hence continuous on $]c', d'[$ since the topology is the order topology. $\square$

In order to prove Lemma 4.3, we introduce the following notations. Set $\Delta(I) = \{(x, y) \in I^2 \mid x < y\}$ and for $* \in \{=, <, >\}$, $\Delta_*(f) := \{(x, y) \in I \mid f(x) * f(y)\}$. Hence to prove Lemma 4.3, one needs to show that there is a subinterval $I' \subset I$ such that $\Delta(I') \subset \Delta_*(f)$ for some $* \in \{=, <, >\}$.

Since $I^2 = \Delta_=(f) \cup \Delta_<(f) \cup \Delta_>(f)$, this is a particular case of the following lemma.

Lemma 4.5 (O-minimal pigeonhole principle)**.** *Let $S_1, \ldots, S_n \subset R^2$ definable such that $I^2 \subset S_1 \cup \cdots \cup S_n$. Then there is some k and a subinterval $I' \subset I$ such that $\Delta(I) \subset S_k$.*

If $A \subset R^{m+n}$, and $x \in R^m$, we denote by $A_x \subset R^n$ the fiber of A over $x :$ $A_x = \{y \in R^n \mid (x, y) \in A\}$, which is definable if A is definable.

Proof. For $k = 1, \ldots, n$, set $A_k := \{x \in I \mid \exists x' > x,]x, x'[\subset (S_k)_x\}$. By hypothesis and o-minimality, there is some k such that A_k contains an interval J. Define the map

$$g : x \in J \mapsto \sup\{x' \in I \mid x' > x \wedge]x, x'[\subset (S_k)_x\} \in R \cup \{+\infty\} .$$

We claim that there is a bounded interval $I' \subset J$ and $d > \sup(I')$ such that $g(x) > d$ for every $x \in I'$.

Assuming this claim, then for any $x, x' \in I'$ such that $x < x'$, we have since $g(x) > d$ that $x' < d$ hence $]x, x'[\subset (S_k)_x$, hence $(x, x') \in S_k$, *i.e.* $\Delta(I') \subset S_k$.

It remains to prove the claim. To this purpose, set $A := \{y \in I \mid \forall x \in J(x < y \to g(x) \leq g(y))\}$.

We have two cases to consider. The first case is if A contains an interval J'. Then g is increasing on J'. Let $c \in J'$. Since for every $x \in J$, $g(x) > x$, we have $g(c) > c$. By density, there are $d, d' \in (c, g(c))$ with $d < d'$. Then $I' := J' \cap (c, d')$ is an open interval and for every $x \in J'$, $g(x) \geq g(c) > d$.

The other case to consider is when $J \backslash A$ contains an interval J' on which g is not increasing. Then for any $c \in J'$, there is some $x_1 \in]\inf J, c[$ such that $g(x_1) > g(c)$. By iterating, we build a sequence $x_1 > x_2 > \cdots > \inf J'$ such that $g(x_{i+1}) > g(x_i) > g(c)$. By o-minimality, there is an interval $I' \subset]\inf J, c[$ such that for every $x \in I'$, $g(x) > d := g(c) > c$, which finishes the proof of the claim, of the lemma, and of the monotonicity theorem. $\square$

4.2 Uniform finiteness

Definition 4.6. If $A \subset R^{m+n}$, say that A is *finite over* R^m if for every $x \in R^m$, A_x is finite. Say that A is *uniformly finite over* R^m if there is some $k \in \mathbb{N}$ such that for every $x \in R^m$, $\#A_x \leq k$.

A key fact about o-minimality is that for definable sets in o-minimal structures, finiteness implies uniform finiteness. For now one we only prove this in the following particular case.

Proposition 4.7 (Uniform finiteness for R^2)**.** *Let $A \subset R^2$ be definable and finite over R. Then A is uniformly finite over R.*

Corollary 4.8. *Given a definable set $A \subset R^2$ that is finite over R, there are points $a_1 < \cdots < a_s$ in R such that the intersection of A with each vertical strip $]a_i, a_{i+1}[\times R$ is the union of graphs $\Gamma(f_1) \cup \cdots \cup \Gamma(f_n)$ where $f_j :]a_i, a_{i+1}[\to R$ is definable continous and $f_1 < \cdots < f_n$.*

Proof. Call a point $(a, b) \in R^2$ good if there is a box $I \times J$ around it such that either $I \times J \cap A = \emptyset$, either $]a, b[\in A$ and $I \times J \cap A = \Gamma(f)$, where f is a continuous map $f : I \to R$ (note that such f is necessarily unique and definable). Also, call a point $(a, +\infty) \in R \times \bar{R}$ good if there is a box $I \times J$ disjoint from A such that $a \in I$ and $J =]b, +\infty[$ for some b. Define goodness for points of the form $(a, -\infty)$ similarly.

Observe that the sets $\{(a, b) \in R^2 \mid (a, b) \text{ is good}\}$, $\{a \in R \mid (a, +\infty) \text{ is good}\}$ and $\{a \in R \mid (a, -\infty) \text{ is good}\}$ are definable.

Now define maps $f_1, \ldots, f_n, \ldots$ as follows. The domain of f_n is the set $\mathrm{dom}(f_n) := \{x \in R \mid \#A_x \geq n\}$ and for $x \in \mathrm{dom}(f_n)$, we set $f_n(x)$ to be the n-th element of A_x. Observe that for fixed n, f_n is definable (possibly with an empty domain).

For each $a \in R$, consider n_a, the maximal $n \geq 0$ such that f_n is defined and continuous on an interval containing a.

Set $\mathcal{B} = \{a \in R \mid a \in \mathrm{cl}(\mathrm{dom}(f_{n_a+1}))\}$ and its complement $\mathcal{G} = \{a \notin R \mid a \in \mathrm{cl}(\mathrm{dom}(f_{n_a+1}))\}$.

Note that it is *a priori* not clear whether $\mathcal{B}$ and $\mathcal{G}$ are definable, since their definition involve a quantification over the parameter $n \in \mathbb{N}$.

Observe that if $a \in \mathcal{G}$, for $n = n_a$ as above, then there is a interval around a such that $f_1, \ldots, f_n$ are defined on it and $\mathrm{dom}(f_{n+1})$ are disjoint from it. In particular, if $a \in \mathcal{G}$, then $\#A_x$ is constant on an interval around a and (a, b) is good for every $b \in \bar{R}$.

We claim that if $a \in \mathcal{B}$, then there is some $b \in \bar{B}$ such that (a, b) is not good. Note that this claim implies that $\mathcal{B}$ and $\mathcal{G}$ are definable.

To prove it, we introduce the following elements of $\bar{R}$, where $n = n_a$ is as above:

$$\lambda(a, -) := \lim_{x \to a^-} f_{n+1}(a)$$

if f_{n+1} is defined in some interval (u, a), $+\infty$ otherwise;

$$\lambda(a, 0) := f_{n+1}(a)$$

if $a \in \mathrm{dom}(f_{n+1})$, $+\infty$ otherwise;

$$\lambda(a, +) := \lim_{x \to a^+} f_{n+1}(a)$$

if f_{n+1} is defined in some interval (a, u), $+\infty$ otherwise.

Such limits exist by Corollary 4.2. Now set $\mu(a) := \min\{\lambda(a, -), \lambda(a, 0), \lambda(a, +)\}$. Then $\mu(a)$ is the least $b \in \bar{R}$ such that (a, b) is not good, proving the claim, hence the definability of $\mathcal{B}$ and $\mathcal{G}$. Observe that this also proves that μ is a definable map.

To finish the proof, observe that if $\mathcal{B}$ is finite, say $\mathcal{B} = \{a_1, \ldots, a_s\}$, with $a_0 = -\infty < a_1 < \cdots < a_s < a_{s+1} = +\infty$, then $\#A_x$ is locally constant on each $]a_i, a_{i+1}[$, hence constant (by the monotonicity theorem for example). Hence $\#A_x$ is bounded globally on R.

Suppose now that $\mathcal{B}$ is infinite. We will derive a contradiction, which will finish the proof.

Consider the (definable) sets

$$\mathcal{B}_- := \{a \in \mathcal{B} \mid \exists y (y < \mu(a), (a, y) \in A)\},$$

$$\mathcal{B}_+ := \{a \in \mathcal{B} \mid \exists y (y > \mu(a), (a, y) \in A)\}$$

and the maps

$$\mu_- : a \in \mathcal{B}_- \mapsto \max\{y \in R \mid y < \mu(a), (a, y) \in A\},$$

$$\mu_+ : a \in \mathcal{B}_+ \mapsto \min\{y \in R \mid y > \mu(a), (a, y) \in A\}.$$

Since $\mathcal{B}_-$ and $\mathcal{B}_+$ are definable and $\mathcal{B}$ is infinite, at least one among the sets $\mathcal{B}_- \cap \mathcal{B}_+$, $\mathcal{B}_+ \backslash \mathcal{B}_-$, $\mathcal{B}_- \backslash \mathcal{B}_+$, $\mathcal{B} \backslash (\mathcal{B}_- \cup \mathcal{B}_+)$ contains an interval. Each of those four cases leads to a contradiction in a similar way. We treat only the case where $\mathcal{B}_+ \backslash \mathcal{B}_-$ contains an interval I. By the monotonicity theorem, up to restricting I we can assume μ and μ_+ are strictly monotone and continuous on I. We also have $\mu < \mu_+$.

Partition I into the two subsets $\{x \in I \mid (x, \mu(x)) \in A\}$ and $\{x \notin I \mid (x, \mu(x)) \in A\}$. At least one of them contains an interval, so we can assume that $\Gamma(\mu_{|I}$ is either contained in A or disjoint from A. By continuity of μ and μ_+ on I, it is clear in both cases that $\Gamma(\mu_{|I}$ contains only good points. Contradiction with the fact that $(a, \mu(a))$ is not good by definition of μ. $\square$

4.3 Cells

If X is a definable set, define $C(X)$ to be the set of continuous definable maps $f : X \to R$ and $C_\infty(X) = C(X) \cup \{-\infty, +\infty\}$ (we view $-\infty$ and $+\infty$ as constants maps). If $f, g \in C_\infty(X)$ satisfy $f < g$, set

$$(f, g)_X := \{(x, y) \in X \times R \mid (f(x) < y < g(x)\}.$$

Definition 4.9. A $(i_1, \ldots, i_n)$-*cell* of R^n, where $i_j \in \{0, 1\}$, is defined by induction on n as follows.

- A (0)-cell of R is a singleton $\{a\} \subset R$ and a (1)-cell of R is an interval $]a, b[\subset R$.
- Assume that $(i_1, \ldots, i_n)$-cells of R^n are defined. Then a $(i_1, \ldots, i_n, 0)$-cell of R^{n+1} is a set of the form $\Gamma(f) \subset R^{n+1}$ for some $f \in C(X)$, where X is a $(i_1, \ldots, i_n)$-cell and a $(i_1, \ldots, i_n, 1)$-cell of R^n is a set of the form $(f, g)_X$ for some $(i_1, \ldots, i_n)$-cell X and $f, g \in C_\infty(X)$ such that $f < g$.

A *cell* is a $(i_1, \dots, i_n)$-cell of R^n, for some $(i_1, \dots, i_n)$. An *open cell* is a $(1, \dots, 1)$-cell.

The terminology is justified by the fact that $(1, \dots, 1)$-cells are open of the order topology. Note that non-open cells are thin, in the sense that a finite union of non-open cell has empty interior.

Lemma 4.10. *(1) Each cell is locally closed,* i.e. *open in its closure.*

(2) Each cell is isomorphic, via a coordinate projection, to an open cell.

(3) Each cell is definably connected.

Proof. By induction on the embedding dimension and Lemma 3.12. □

Definition 4.11. A *cell decomposition* of R^n is defined inductively as follows. A cell decomposition of R is a finite partition of R into cells. A cell decomposition of R^{n+1} is a finite partition of R^{n+1} into cells A such that the set of $\pi(A)$ form a cell decomposition of R^n, where $\pi : R^{n+1} \to R$ is the coordinate projection.

The following cell decomposition is central in the theory of o-minimal structures. In the following strong form, it is due to Knight, Pillay and Steinhorn [KPS86].

Theorem 4.12 (Cell decomposition)**.** *For every $m \geq 1$, the following holds.*

(CD_m) *Given definable sets $A_1, \dots, A_k \subset R^m$, then there is a cell decomposition of R^m such that for each $i = 1, \dots, k$, each cell is either included in A_i or disjoint from A_k.*

(PC_m) *If $f : A \subset R^m \to R$ is a definable map, then there exists a cell decomposition of R^m partitioning A such that f is continuous on each of the cell included in A.*

(UF_m) *If a definable set $A \subset R^m$ is finite over R^{m-1}, then A is uniformly finite over R^{m-1}.*

Remark 4.13. (1) We say that the cell decomposition of (CD_m) is a cell decomposition of $A_1, \dots, A_k$, or that it is *adapted* to $A_1, \dots, A_k$. We say that the cell decomposition of (PC_m) is *adapted to f*.

(2) Note that (CD_1) is the definition of o-minimality, (PC_1) is the monotonicity theorem 4.1 and (UF_2) is Proposition 4.7.

(3) Observe that (CD_m) implies (UF_m). The proof goes by induction. Assuming (CD_i) and (PC_i) holds for each $i \leq m$, one first show (UF_{m+1}), which is the core of the proof, then (CD_{m+1}) and (PC_{m+1}).

Before proving the cell decomposition theorem, we state and prove some important consequences.

Theorem 4.14. *If $\mathcal{R} = (R, <, \dots)$ is an o-minimal structure, then $\mathcal{R}$ is strongly o-minimal,* i.e. *any model of* $\mathrm{Th}(\mathcal{R})$ *is o-minimal.*

Remark 4.15. This theorem "explains" the strength of o-minimaly. For comparison, one defines a *minimal* (resp. *strongly minimal*) structure $\mathcal{M} = (M, \dots)$ as any structure such that the definable (with parameters) subsets of M are finite and cofinite sets (resp. ...). While strongly minimal theories (*e.g.* ACF) enjoy many good properties, it is not true that minimal implies strongly minimal (*e.g.* $(\mathbb{N}, <)$).

Proof. Let $\mathcal{R} = (R, <, \dots)$ be an o-minimal $\mathcal{L}$-structure. Let $\mathcal{R}' = (R', <, \dots)$ be a model of $\mathrm{Th}(\mathcal{R})$, and $\varphi(x, y)$ an $\mathcal{L}$-formula, where x is a single variable, and y a tuple of m variables. Let $\psi(x, y)$ an $\mathcal{L}$-formula such that for $b \in R^m$, $\psi(\mathcal{R}, b)$ is the boundary of $\varphi(\mathcal{R}, b)$. Note that φ describes also the boundary in $\mathcal{R}'$. In view of Remark 3.9, it is enough to prove that for every $b \in R'^m$ such that $\varphi(\mathcal{R}', b)$ is neither empty nor R', $\psi(\mathcal{R}', b)$ is a non-empty finite set. By o-minimality and uniform finiteness (UF_{m+1}), there is some integer k such that $\#\psi(\mathcal{R}, b) \le k$ for every $b \in R^m$. Then the $\mathcal{L}$-sentence

$$\forall b (\exists x(\varphi(x, b)) \wedge \exists x(\neg\varphi(x, b))) \to (1 \le \# \{x \mid \varphi(x, b)\} \le k)$$

is true in $\mathcal{R}$, hence in $\mathcal{R}'$, which finishes the proof. □

Exercise 4.16. Note that we mainly used in the previous proof the uniform finiteness property. Reciprocally, show directly that if a structure is strongly o-minimal, then it satisfies the uniform finiteness property (UF_m). (Hint : use the compactness theorem.)

Proposition 4.17. *(1) Every definable set $X \subset R^n$ admits a finite number of definably connected components, which form a partition of X and are open and closed in X.*

(2) Let $X \subset R^{n+m}$ be definable. Then there exists a $k \in \mathbb{N}$ such that for every $x \in R^n$, X_x admits at most k definable connected components.

(3) Let $\mathfrak{R} = (\mathbb{R}, <, \dots)$ be an o-minimal expansion of the ordered sets of reals numbers. Then for $X \subset \mathbb{R}^n$ definable in $\mathfrak{R}$, we have that X is definably connected if and only if X is connected in the usual topological sense.

Proof. (Sketch)

(1) If $\{C_1, \dots, C_k\}$ is a partition of X into cells, then for every $I \subset \{1, \dots, k\}$, set $C_I = \bigcup_{i \in I} C_i$ and consider maximal (for the inclusion) I_0 such that C_{I_0} is definably connected. Then C_{I_0} is a definably connected component of X.

(2) Using the proof of (1), the number of cells in a cell decomposition of X gives a bound for the number of definably connected components of X_x, for every $x \in R^n$.

(3) It is clear that connected implies definably connected. To show the contrary, it is enough to prove it for cells, in which case the result is proven by induction.

□

4.4 Proof of the cell decomposition theorem

To prove the cell decomposition theorem, we assume by induction that it holds for k up to m, and show successively that (UF_{m+1}), (CD_{m+1}) and (PC_{m+1}) holds.

Proof of (UF_{m+1}). We fix $Y \subset R^{m+1}$ definable and finite over R^m.

Call a box $B \subset R^m$ Y*-good* if for every $(x, y) \in Y$ such that $x \in B$, there is an interval I around y such that $B \times I \cap Y$ is the graph of a continuous map $f : B \to R$. Note that such f is automatically definable. Call a point $x \in R^m$ Y*-good* if there is a box B around x that is good. Note that the set of good points is definable.

Claim 1: Suppose that the box $B \subset R^m$ is Y-good. Then there are definable continuous functions $f_1 < \cdots < f_k : B \to R$ such that $Y \times B \cap R = \Gamma(f_1) \cup \ldots \Gamma(f_k)$.

Proof: exercise (Hint : first show the result locally around a point $x \in B$, then use the fact that B is definably connected).

Claim 2: If $A \subset R^m$ is definably connected and all points of A are Y-good, then there are definable continuous functions $f_1 < \cdots < f_k : B \to R$ such that $Y \times A \cap R = \Gamma(f_1) \cup \ldots \Gamma(f_k)$.

Proof: Clear using claim 1 and definable connectedness of A.

Claim 3: Each open cell in R^m contains a good point.

Proof: It is enough to show the claim for a box $B = B' \times]a, b[$. For each $x \in B'$, consider the definable set

$$Y(x) := \left\{(r, s) \in R^2 \mid a < r < b, (x, r, s) \in Y\right\}.$$

It is finite over R, hence by Corollary 4.8, the set of points $r \in R$ such that r is not $Y(x)$-good is finite. Hence the set

$$\mathrm{Bad}(Y) := \{(x, r) \in B \mid r \text{ is not } Y(x)\text{-good}\}$$

has no interior point. By inductive assumption (CD_m), there is a cell decomposition of R^m adapted to B and $\mathrm{Bad}(Y)$. Take an open cell C of this partition such that $C \subset B$. Then $C \cap \mathrm{Bad}(Y) = \emptyset$. We can replace B by C, hence assume that $\mathrm{Bad}(Y) = \emptyset$. For each $x \in B'$, we can then apply Claim 2 to $Y(x)$ and obtain an integer $k(x)$ such that $\#Y_{(x,r)} = k(x)$ for every $r \in]a, b[$

We need to show that there is a bound for the numbers $k(x)$, with $x \in B'$. Fix $r_0 \in]a, b[$ and set

$$Y^{r_0} := \{(x, s) \in R^m \mid (x, r_0, s) \in Y\},$$

which is definable and finite over R^{m-1}. By the induction hypothesis (UF_m), there is some $N \geq 0$ such that $\#Y^{r_0}_x \leq N$ for every $x \in B'$. Hence for every $x \in B'$, $k(x) \leq N$. In particular, for every $(x, r) \in B$, $\#Y_{x,r} \leq N$.

For $i = 1, \ldots, N$, consider $B_i = \{x \in B \mid \#Y_x = N\}$ and define maps $f_{i1} < \cdots < f_{ii}$ on B_i by the condition $Y_x = \{f_{i1}(x), \ldots, f_{ii}(x)\}$. All the f_{ij} are definable. By the induction hypothesis (PC_m) applied to each f_{ij} successively, and (CD_m) to find a common refinement, we find a cell decomposition of R^m such that for each of its cells C, if $C \subset B_i$, then f_{ij} is continuous on C. Since B is open, there is at least one open cell C of the decomposition contained in B_i for some i. By construction, all points of C are Y-good, proving Claim 3.

We now finish the proof as follows. Consider a cell decomposition of R^m adapted to the (definable) set of Y-good points, and consider one of its open cells C. By Claim 3, C contains an Y-good point, hence every point of C is Y-good. By Claim 2, there is some $N_C \geq 0$ such that $\#Y_x \leq N_C$ for every $x \in C$. If C is a non-open cell, using Lemma 4.10 (2) and the induction hypothesis, we also have a similar N_C. Since there are only finitely many cells in the decomposition, for $N := \max N_C$, we have that $\#Y_x \leq N$ for every $x \in R^m$. □

In order to prove (CD$_{m+1}$), recall from Lemma 3.8 that the boundary ∂A of a definable set $A \subset R$ is a finite set of points, and that every interval between two successive points of ∂A is either included in A or disjoint from A. For $A \subset R^{m+1}$, we introduce the relative boundary as follows :

$$\partial_m A := \{(x, y) \in R^m \mid y \in \partial A_x\}.$$

Observe that if A definable, then $\partial_m A$ is definable and finite over R^m, hence we can apply (UF$_{m+1}$) to it.

Proof of (CD$_{m+1}$). Set $Y = \partial_m A_1 \cup \cdots \cup \partial_m A_k$. Since $Y \subset R^{m+1}$ is definable and finite over R^m, by (UF$_{m+1}$) there is some N such that $\#Y_x \leq N$ for every $x \in R^m$. For $i = 1, \ldots, N$, set $B_i = \{x \in R^m \mid \#Y_x = i\}$ and define maps $f_{i1} < \cdots < f_{ii}$ on B_i such that $Y_x = \{f_{i1}(x), \ldots, f_{ii}(x)\}$. Also put $f_{i0} = -\infty$, $f_{ii+1} = +\infty$. Finally define

$$C_{l,i,j} = \{x \in B_i \mid f_{ij}(x) \in (A_l)_x\}$$

and

$$D_{l,i,j} = \{x \in B_i \mid]f_{ij}(x), f_{ij+1}(x)[\subset (A_l)_x\}.$$

Using inductive hypothesis (CD$_m$) and (PC$_m$), take a cell decomposition of R^m partitioning all $C_{l,i,j}$ and $D_{l,i,j}$ and such that each map f_{ij} is continuous on each of the cells. Then for each cell C, consider the partition of $C \times R$

$$\{(f_{i0}, f_{i1})_C, \ldots, (f_{ii}, f_{ii+1})_C, \Gamma(f_{i1}|C), \ldots, \Gamma(f_{ii}|C)\},$$

where i is such that $C \subset B_i$. The union of all such partitions for varying C provides the required cell decomposition of R^{m+1}. □

To prove (PC$_{m+1}$), we will need the following elementary lemma, which proof is left as an exercise.

Lemma 4.18. *Let X be a topological space, $(R_1, <)$ and $(R_2, <)$ two dense total orders without endpoints and $f : X \times R_1 \to R_2$ a map such that for each $(x, r) \in X \times R_1$, $f(x, \cdot)$ is continuous and monotone and $f(\cdot, r)$ is continuous. Then f is continuous.*

Proof of (PC$_{m+1}$). Let $f : A \subset R^{m+1} \to R$ be a definable map. We need to find a cell decomposition adapted to A such that f is continuous on each of its cells. Using (CD$_{m+1}$), it is enough to find a finite partition of A into definable sets A_i such that f is continuous on each A_i. Similarly, we can assume that A is already a cell.

If A is not open, by Lemma 4.10, there is a coordinate projection $\pi : A \to \pi(A) \subset R^d$ which is bijective. By (PC$_d$), there is a finite definable partition of $\pi(A)$ such that $f \circ \pi^{-1}$ is continuous on each of the pieces, which induce the required partition of A.

If A is open, we proceed as follows. Call f *well-behaved* at $(x, r) \in R^{m+1}$ if (x, r) is contained in a box $B \times]a, b[$ such that $B \times]a, b[\subset A$, the map $f(x, \cdot)$ is continuous and monotone on $]a, b[$, the map $f(\cdot, r)$ is continuous at x. Note that the set WB of well-behaved points is definable.

Claim: the set WB is dense in A.

To show this, one must show that every box $B \times]a, b[$ contained in A intersects WB. By the monotonicity theorem, for every $x \in B$ there is a maximal $\lambda(x) \in [a, b]$ such that the map $f(x, \cdot)$ is continuous and monotone on $]a, \lambda(x)[$. Since $\lambda : B \subset R^m \to R$ is definable, by (PC_m) there is a box $C \subset B$ on which λ is continuous. Up to restricting C, we can then assume that there is some $b' \in R$ such that $a < b' < \lambda(x)$ for every $x \in C$. Choose some $r \in]a, b'[$. By (PC_m), up to restricting C we can assume that $f(\cdot, r)$ is continuous on C, showing that f is well-behaved at any (x, r) with $x \in C$. This finishes the proof of the claim.

Take a cell decomposition of R^m adapted to A and WB. Pick an open cell D contained in A. We need to show that f is continuous on D. Since by the claim D intersects WB, D is included in WB. Hence D is a union of boxes where f satisfies the conditions of Lemma 4.18, hence f is continuous on D, which finishes the proof of (PC_{m+1}) and of the cell decomposition theorem. $\square$

4.5 Dimension theory

We develop here a dimension theory for definable sets in o-minimal structures.

Definition 4.19. The *dimension* $\dim(X)$ of a non-empty definable set $X \subset R^m$ is

$$\dim(X) := \max\{i_1 + \cdots + i_m \mid X \text{ contains an } (i_1, \ldots, i_m) - \text{cell}\}.$$

The dimension of the empty set is $-\infty$.

Note that the dimension of $X \subset R^m$ is m if and only if X contains an open cell. By cell decomposition, we have that $\dim(X) = 0$ if and only if X is finite.

Recall from Lemma 4.10 that a $(i_1, \ldots, i_n)$-cell $C \subset R^n$ is isomorphic, via a coordinate projection, to an open cell of R^d where $d = i_1 + \cdots + i_n$. We denote it by $\pi_C : C \to \pi_C(C) \subset R^d$. It results from property (2) of the following proposition that such a cell is indeed of dimension d.

Proposition 4.20. *(1) If $X \subset Y \subset R^m$ are definable, then* $\dim(X) \leq \dim(Y)$.

(2) If $X \subset R^n$ and $Y \subset R^m$ are definable, and $f : X \to Y$ is a definable bijection, then $\dim(X) = \dim(Y)$.

(3) If $X, Y \subset R^m$, then $\dim(X \cup Y) = \max\{\dim(X), \dim(Y)\}$.

(4) If $X \subset R^n$ is definable and non-empty, then $\dim(X)$ is the biggest d such that there is a coordinate projection $\pi : R^n \to R^d$ and $\pi(X)$ contains an open cell.

(5) If $X \subset R^n$ is definable, and $f : X \to R^m$ is a definable map, then $\dim(f(X)) \leq \dim(X)$.

(6) If $X \subset R^n$ and $Y \subset R^m$ are definable, then $\dim(X \times Y) = \dim(X) + \dim(Y)$.

(7) *If $X \subset R^n$ is definable, then* $\dim(\partial X) < \dim(X)$*, in particular,* $\dim(\mathrm{cl}(X)) = \dim(X)$*.*

We first prove the following lemma :

Lemma 4.21. *Let $C \subset R^m$ be an open cell and $f : C \to R^m$ an injective definable map. Then* $\mathrm{int}(f(C)) \neq \emptyset$*.*

Proof. By induction on m. The case $m = 1$ is clear, since $f(C)$ is infinite. Fix $m > 1$ and assume the result is known for $i < m$. By cell decomposition, $f(C) = A_1 \cup \ldots A_k$, where A_i are cells. Since C is open, by the cell decomposition theorem there is some i such that $f^{-1}(A_i)$ has non-empty interior open, *i.e.* contains a box B. By (PC), we can assume that f is continuous on B. We claim that C_i is open. If not, we can find a definable homeomorphism $\pi : C_i \to C'_i \subset R^{m-1}$ where C'_i is a cell. Set $g = \pi \circ f_{|B}$. Then g is a definable injective continuous map from B to R^{m-1}. Write $B = B' \times]a, b[$. For any $c \in]a, b[$, the map $g(\cdot, c)$ is a definable injection from an open cell of R^{m-1} to R^{m-1}, hence its image has non-empty interior. Say it contains a box D. Fix $y \in D$ and choose $x \in B'$ such that $g(x, c) = y$. By continuity of g, if $c' \neq c$ is close enough to c, then $g(x, c')$ is in D, contradicting the injectivity of g. □

Proof of 4.20. (1) is clear. For (2), if $f : X \subset R^n \to Y \subset R^m$ is a definable bijection, and $d = \dim(X)$, $e = \dim(Y)$, it is enough to prove $d \leq e$. Let $C \subset X$ be a cell of dimension d, and $D \subset R^d$ an open cell such that there is a coordinate projection such that its restriction to C is a bijection $\pi_C : C \to D$. Up to replacing Y by $f(C)$, we can assume that X is an open cell, *i.e.* $d = n$. Let $Y = Y_1 \cup \cdots \cup Y_s$ a decomposition of Y into cells. There is some i such that $f^{-1}(Y_i)$ contains an open cell C. Let $r = \dim(Y_i) \leq e$. It is enough to show that $n \leq r$. If $n > r$, denoting by π_{Y_i} a coordinate projection of Y_i into an open cell of R^r, we have that $g := \pi_{Y_i} \circ f_{|C}$ is an injection from C to R^r Viewing R^r as closely embedded into R^n, from Lemma 4.21, $g(C)$ has non-empty interior in R^n, contradiction.

For (3), set $d = \dim(X \cup Y)$ and pick $C \subset X \cup Y$ a d-dimensional cell. Then $\pi_C(C \cap X) \cup \pi_C(C \cap Y)$ contains an open cell. By cell decomposition, one of $\pi_C(C \cap X)$ or $\pi_C(C \cap Y)$ contains an open cell. Say, without loss of generality, that $\pi_C(C \cap X)$ contains an open cell D. By (2), since π_C is a bijection, $\dim(\pi_C^{-1}(D)) = \dim(D) = d$, and $\pi_C^{-1}(D) \subset X$, hence $d \leq \dim(X)$.

Property (4) is clear for cells and follows in general by cell decomposition. □

Before finishing the proof of the proposition, we need to show the following.

Proposition 4.22. *Let $X \subset R^{m+n}$ be definable. For $d \in \{-\infty, 0, 1, \ldots, n\}$, set $X(d) = \{x \in R^m \mid \dim(X_x) = d\}$. Then $X(d)$ is definable and*

$$\dim(X \cap X(d) \times R^n) = d + \dim(X(d)).$$

In particular, $\dim(X) = \max\{d + \dim(X(d)) \mid d = 0, \ldots, n\}$.

Proof. (Sketch) Observe (or rather prove by induction on n) that if X is a $(i_1, \ldots, i_{m+n})$-cell, then it projection $\pi(X) \subset R^m$ is a $(i_1, \ldots, i_m)$-cell and for every $x \in \pi(X)$, X_x is a $(i_{m+1}, \ldots, i_{m+n})$-cell. Hence the result holds for cells.

For X an arbitrary definable set, take a cell decomposition adapted to X. Then its projection to R^m is a cell decomposition of R^m, and one can check that it is adapted to each $S(d)$. In particular, each $S(d)$ is a finite union of cells, hence definable. The conclusion now follows, by using the result for cells. □

End of the proof of Proposition 4.20. Property (5) follows from Proposition 4.22 applied to the graph of f. Property (6) also, by applying it to $X \times Y$.

It remains to prove (7), *i.e.* that if $X \subset R^n$ is definable, then $\dim(\partial X) < \dim(X)$. We work by induction on n. For $n = 1$, it is the definition of o-minimality. Fix $n > 1$ and assume the result holds for integers up to $n-1$. Let $\pi : R^n \to R$ the projection to the first coordinate.

Step 1: For $A, B \subset R^n$ definable. If for every $x \in R$, $\dim(A_x) < \dim(B_x)$, then $\dim(A) < \dim(B)$.

It is clear if A and B are cells, and one reduces to this case by considering a cell decomposition adapted to A,B,$A(d)$, $B(d)$ for $d = 0, 1, \dots, n$

Step 2: Assume that for every $x \in R$, $\partial X_x = (\partial X)_x$. Then we have $\dim(\partial X_x) = \dim((\partial X)_x) < \dim(X_x)$ by induction hypothesis. Hence we are done by Step 1.

Step 3: The set $S = \{x \in R \mid \partial X_x = (\partial X)_x\}$ is finite.

To show this, observe that $S = \{x \in R \mid \mathrm{cl}(X_x) = (\mathrm{cl}(X))_x\}$, and that for every $x \in R$, $\mathrm{cl}(X_x) \subset (\mathrm{cl}(X))_x$.

Set $B = \left\{(a, b) \in R^{2(n-1)} \mid a_i < b_i\right\}$. Each $z \in B$ defines a box $B(z) \subset R^{n-1}$. Set

$$T = \{(x, z) \in R \times B\} \mid \mathrm{cl}(X)_x \cap B(z) \neq \emptyset = \mathrm{cl}(X_x) \cap B(z).$$

Then one can check that for every $x \in S$, $\mathrm{int}(T)_x \neq \emptyset$ but for every $z \in B$, $\mathrm{int}(T)_z = \emptyset$. By Proposition 4.22, since $\dim(T_x) = 2(n-1)$ for every $x \in S$, $\dim(T) = 2(n-1) + \dim(S)$. But since $\dim(T_z < 1$ for every $z \in B$, $\dim(T) < 2(n-1) + 1$, hence $\dim(S) < 1$, *i.e.* S is finite.

Step 4: Since S is finite, we can finish the proof by applying Step 2 to $X \cap (R \backslash S) \times R^{n-1}$, which has the same dimension than X. □

4.6 O-minimal expansions of fields

We now work in a structure $\mathfrak{R} = (R, <, 0, 1, +, -, \cdot, \dots)$ which is an o-minimal expansion of an ordered field.

Proposition 4.23. *R is a real closed field.*

Proof. One needs to check that any odd degree polynomial with coefficients in R admits a root in R. This follows from the intermediate value theorem proved in Lemma 3.12. □

We can consider the absolute value $|x|$ of $x \in R$, and the usual Euclidian norm $\|x\| := \sqrt{x \cdot x}$ (for $x \in R^n$), which are both definable.

Let $X \subset R$ be definable and non-empty. If X has a least element, we define $e(X)$ to be the least element of X. Otherwise, let $]a, b[$ be its "left-most" interval, *i.e*

$a = \inf(X)$ and $b = \sup\{x \in X \mid]a, x[\subset X\}$. Now define

$$e(X) := \begin{cases} 0 & \text{if } a = -\infty, b = +\infty \\ a+1 & \text{if } a \in R, b = +\infty \\ b-1 & \text{if } a = -\infty, b \in R \\ \frac{a+b}{2} & \text{if } a, b \in R \end{cases}$$

Proposition 4.24 (Definable choice)**.** *Let $X \subset R^{m+n}$. Consider the coordinate projection $\pi : X \subset R^{m+n} \to R^m$. Then π admits a definable section,* i.e., *there is a definable map $f : \pi(X) \to R^n$ such that for every $x \in \pi(X)$, $(x, f(x)) \in X$.*

Proof. By induction on n, we can assume that $n = 1$. Then for $x \in \pi(X)$, one defines $f(x) := e(X_x)$, which is indeed a definable section of π. □

Corollary 4.25 (Curve selection)**.** *If $a \in \mathrm{cl}(X) \backslash X$, where $X \subset R^n$ is definable, then there exists a definable continuous injective map $f :]0, \varepsilon[\to X$ such that* $\lim_{x \to 0^+} f(x) = a$.

Remark 4.26. For Proposition 4.24 and Corollary 4.25, one does not need multiplication, it is enough to work in an o-minimal ordered group.

Proof. Consider the definable set $\{(t, x) \in R \times X \mid |a - x| = t\}$. By hypothesis, its projection to R contains elements arbitrary small, hence an interval of the form $]0, \varepsilon[$. By definable choice, its admits a definable section $f :]0, \varepsilon[\to X$. By construction, $\lim_{x \to 0^+} f(x) = a$. By the monotonicity theorem, up to restricting ε we can assume that f is continuous and injective on $]0, \varepsilon[$. □

Since we now work in a field, we can differentiate maps, and the set of points where a definable map is differentiable is definable, as well as the differential of a map. Many results obtained previously with continuous definable maps will extend with differentiable maps and even definable maps of class $\mathcal{C}^{(k)}$.

Proposition 4.27 (Rolle)**.** *For $a < b \in R$, let $f : [a, b] \to R$ a continuous definable map and differentiable on $]a, b[$. Then for some $c \in]a, b[$, $f(b) - f(a) = (b - a) f'(c)$.*

Proof. One can assume that $f(a) = f(b)$, in which case one can take any $c \in]a, b[$ such that $f(c)$ is maximal or minimal. □

Corollary 4.28. *Under the hypothesis of the proposition, assume moreover that $f'(x) = 0$ for every $x \in]a, b[$. Then f is constant on $[a, b]$.*

Proposition 4.29. *If $f : I \subset R \to R$ is definable on an interval I, then f is differentiable on all but finitely many points of I.*

Recall that a map f is of class $\mathcal{C}^{(k)}$ if it is k-times differentiable and its k-differential is continuous.

Corollary 4.30. *If $f : I \subset R \to R$ is definable on an interval I, then there is a finite set of points $C \subset I$ such that f is of class $\mathcal{C}^{(k)}$ on $I \backslash C$.*

The proposition follows from the following three lemmas, which proofs are left as exercises, and the monotonicity theorem.

Lemma 4.31. *For each $x \in I$, the limits $f'(x^+) := \lim_{t\to 0^+} \frac{f(x+t)-f(x)}{t}$ and $f'(x^-) := \lim_{t\to 0^-} \frac{f(x+t)-f(x)}{t}$ exists in $\bar{R}$. Moreover, if f is continuous and $f'(x^+) > 0$ for every $x \in I$, then f is strictly increasing and its inverse satisfies ${f^{-1}}'(y^+) = 1/f'(x^+)$ for every $y = f(x)$, $x \in I$ (with by convention $1/{+\infty} = 0$).*

Lemma 4.32. *Let $f : I \to R$ be definable and continuous. Assume that $x \mapsto f'(x^+)$ and $x \mapsto f'(x^-)$ are both R-valued and continuous on I. Then f is differentiable on I and f' is continuous on I.*

Lemma 4.33. *Let $f : I \to R$ be definable. There are only finitely many $x \in I$ such that $f'(x^+) \in \{+\infty, -\infty\}$.*

Cells are defined using continuous functions as boundary maps. We define similarly a $\mathcal{C}^k$-cell by requiring definable maps of class $\mathcal{C}^{(k)}$ as boundaries. In what follows, we call a definable map $f : A \subset R^n \to R$ of class $\mathcal{C}^{(k)}$, for possibly non-open A, if there is an definable open $U \subset R^n$, containing A and a definable map $F : U \to R$ of class $\mathcal{C}^{(k)}$ such that $F_{|A} = f$.

If X is a definable set and $k \in \mathbb{N}$, define $C^k(X)$ to be the set of definable maps $f : X \to R$ and $C^k_\infty(X) = C^k(X) \cup \{-\infty, +\infty\}$ (we view $-\infty$ and $+\infty$ as constants maps). If $f, g \in C^k_\infty(X)$ satisfy $f < g$, set

$$(f, g)_X := \{(x, y) \in X \times R \mid (f(x) < y < g(x)\}\,.$$

Definition 4.34. A $\mathcal{C}^{(k)}$-*cell* of R^n is defined by induction on n as follows:

- The $\mathcal{C}^{(k)}$-cells of R are singletons and intervals.
- A $\mathcal{C}^{(k)}$-cell of R^{n+1} is a set of the form $\Gamma(f) \subset R^{n+1}$ for some $f \in C^k(X)$ or a set of the form $(f, g)_X$ for some $f, g \in C^k_\infty(X)$ such that $f < g$ and X a $\mathcal{C}^{(k)}$-cell of R^n.

Definition 4.35. A $\mathcal{C}^{(k)}$-cell decomposition of R^n is a cell decomposition of R^n into cells that are $\mathcal{C}^{(k)}$-cells.

Then one obtains an analog of the cell decomposition theorem :

Theorem 4.36 ($\mathcal{C}^{(k)}$-cell decomposition)**.** *For every $m, k \geq 1$, the following holds.*

(CD_m) *Given definable sets $A_1, \ldots, A_s \subset R^m$, then there is a $\mathcal{C}^{(k)}$-cell decomposition of R^m such that for each $i = 1, \ldots, s$, each cell is either included in A_i or disjoint from A_i.*

(PD_m) *If $f : A \subset R^m \to R$ is a definable map, then there exists a $\mathcal{C}^{(k)}$-cell decomposition of R^m partitioning A such that f is of class $\mathcal{C}^{(k)}$ on each of the cells included in A.*

Proof. (idea) The proof is by induction on k and m. For $k = 0$ it is the usual cell decomposition theorem. Fix $k \geq 1$ and assume the result holds for $k - 1$. Now for $m = 1$, (CD_1) is o-minimality and (PD_1) is Corollary 4.30. Assuming (CD_i) and (PD_i) for every $i \leq m$, one then show (CD_{m+1}) by using the usual cell decomposition, then applying (PD_m) to the boundary maps obtained. In order to prove (PD_{m+1}), one first show that the set of interior points of A where all k-partial derivatives are defined is dense in A. Then one gets the result on the complement of this set by induction, and on this set by the usual cell decomposition. See [van98, Chapter 7] for details. □

Remark 4.37. In many o-minimal structures of interest, such as $\mathbb{R}_{\mathrm{an,exp}}$, one can prove a cell decomposition theorem with $\mathcal{C}^\infty$ maps, and even real analytic maps as boundaries. However, it is not true in general. There are examples of o-minimal structures where no such analytic cell decomposition holds (by Rolin, Speisseger and Wilkie [RSW03]) but also no $\mathcal{C}^\infty$-cell decomposition (by Le Gal and Rolin [LGR09]).

5 Pila-Wilkie counting theorem

We state and prove in this section the Pila-Wilkie counting theorem, which is the key for most of the diophantine applications of o-minimality.

5.1 Points of bounded height

For $x \in \mathbb{Q}$, say $x = \frac{p}{q}$, $p, q \in \mathbb{Z}$ in lowest terms, the (naive) *height* of x is $H(x) := \max\{|p|, |q|\}$. For $x = (x_1, \ldots, x_n) \in \mathbb{Q}^n$, the height of x is $H(x) := \max\{H(x_1), \ldots, H(x_n)\}$.

For any $X \subset \mathbb{R}^n$, set $X(\mathbb{Q}, H) := \{x \in X \cap \mathbb{Q}^n \mid H(x) \leq H\}$ and observe that it is a finite set.

Recall that a semi-algebraic subset of $\mathbb{R}^n$ is a definable set in the structure $(\mathbb{R}, <, +, -, \cdot)$. For any $X \subset \mathbb{R}^n$, define X^{alg}, the *algebraic part* of X, to be the union of infinite, connected semi-algebraic subsets of X. Note that since it may be an infinite union, X^{alg} is in general not definable, even if X is definable. It may happen that $X^{\mathrm{alg}} = X$ (e.g. if X is itself a connected semi-algebraic set) or $X^{\mathrm{alg}} = \emptyset$ (e.g. if X is a transcendental curve). Define the *transcendental part* of X to be $X^{\mathrm{tran}} := X \setminus X^{\mathrm{alg}}$.

Example 5.1. (1) The set $X = \{(x, y, z) \in \mathbb{R}^3 \mid x > 0, z = x^y\}$ is definable in $\mathbb{R}_{\mathrm{exp}}$. Its algebraic part consists of the infinite union of curves of the form $z = x^r$, where $r \in \mathbb{Q}$.

(2) The set $Y = \{(x, y) \in R^2 \mid x \in]0, 1[, 0 < y < e^x\}$ is definable in $\mathbb{R}_{\mathrm{an}}$, not semi-algebraic, but $Y^{\mathrm{alg}} = Y$.

The Pila-Wilkie counting theorem of [PW06] asserts that there are very few points of bounded height in the transcendental part of a definable set.

Theorem 5.2 (Pila-Wilkie counting theorem)**.** *Let $X \subset \mathbb{R}^n$ be a set definable in an o-minimal expansion of $(\mathbb{R}, <, +, -, \cdot)$. Then for every $\varepsilon > 0$, there is a constant $C = C(X, \varepsilon)$ such that for every $H \geq 1$,*

$$\#X^{\mathrm{tran}}(\mathbb{Q}, H) \leq CH^\varepsilon.$$

Remark 5.3. (1) The term H^ε cannot be improved in general, but it is conjectured by Wilkie that for $\mathbb{R}_{\exp}$, it can be replaced by $\log(H)^\alpha$ for some $\alpha \geq 0$. This has been shown for sets defined using only the restricted exponential by Binyamini-Novikov [BN17b]

(2) We will follow the original proof of Pila and Wilkie [PW06], also presented by Scanlon in [Sca17]. There is an other approach, due to Binyamini-Novikov [BN17a], [BN17b] [BN19], that uses a complex analog of the cell decomposition theorem for real o-minimal structures.

We will in fact prove a uniform version, for definable families.

One can wonder if we can require in the theorem that there exists a semi-algebraic set $X_\varepsilon \subset X$ such that $\#X \backslash X_\varepsilon(\mathbb{Q}, H) \leq CH^\varepsilon$ for every H. This is not true, as shown by the set Y of Example 5.1. However, if we allow X_ε to be definable and contained in X^{alg}, this is possible.

Theorem 5.4 (Pila-Wilkie counting theorem, general version)**.** *Let $X \subset \mathbb{R}^{n+m}$ be a set definable in an o-minimal expansion of $(\mathbb{R}, <, +, -, \cdot)$. Then for every $\varepsilon > 0$, there is a constant $C = C(X, \varepsilon)$ and a definable set $W = W(X, \varepsilon) \subset X$ with the following properties. For every $y \in \mathbb{R}^m$, $W_y \subset (X_y)^{\mathrm{alg}}$ and for every $H \geq 1$,*

$$\#(X_y \backslash W_y)(\mathbb{Q}, H) \leq CH^\varepsilon.$$

The proof rely on the following two theorems, that we will prove in the following weeks.

The main input for the diophantine part of the argument is the following result, about rational points on sets parametrized by functions with small derivatives. Its idea goes back to Bombieri-Pila [BP89] and it apprears in Pila [Pil04].

Theorem 5.5. *Let $k, n \in \mathbb{N}$, with $k < n$. There are for each $d \in \mathbb{N}^*$ an integer $r = r(k, n, d)$ and positive constants $\varepsilon(k, n, d)$ and $C(k, n, d)$ with the following properties. Assume that $\varphi :]0, 1[^k \subset \mathbb{R}^k \to \mathbb{R}^n$ is a function of class $\mathcal{C}^{(r)}$, with $\left|\varphi^{(\alpha)}(x)\right| \leq 1$ for all $x \in]0, 1[^k$ and $\alpha \in \mathbb{N}^k$ with $|\alpha| \leq r$. Set $X = \varphi(]0, 1[^k)$. Then for every $H \geq 1$, the set $X(\mathbb{Q}, H)$ is contained in at most*

$$C(k, n, d)H^{\varepsilon(k,n,d)}$$

hypersurfaces of degree at most d. Moreover, $\varepsilon(k, n, d) \to 0$ as $d \to +\infty$.

In order to use the previous theorem, we need to show that we can apply it to definable sets. Although we are ultimately interested in the field $\mathbb{R}$, we need to work in a general o-minimal structure $\mathfrak{R} = (R, <, +, -, \cdot)$.

Definition 5.6. (1) Say that a set $X \subset R^m$ is *strongly bounded* if there is some $n \in \mathbb{N}$ such that $X \subset [-N, N]^m$. Say that a map is strongly bounded if its graph is.

(2) Let $X \subset R^m$ be definable and $d = \dim(X)$. A definable map $\varphi :]0, 1[^d \to X$ is called a *partial parametrization* of X. A finite set S of partial parametrization of X is called a *parametrization* of X if $\bigcup_{\varphi \in S} \mathrm{Im}(\varphi) = X$.

(3) A parametrization S of a definable set X is called an *r-parametrization* if every $\varphi \in S$ is of class $\mathcal{C}^{(r)}$ and has the property that $\varphi^{(\alpha)}$ is strongly bounded for every $\alpha \in \mathbb{N}^k$ with $|\alpha| \leq r$.

(4) An r-parametrization is called a *strong r-parametrization* if for every $\varphi \in S$, $\varphi^{(\alpha)}$ is bounded by 1 for every $\alpha \in \mathbb{N}^k$ with $|\alpha| \leq r$.

(5) Let S be an r-parametrization of a definable set $X \subset R^m$ and $F : X \to R^n$ a definable map. Then we say that S is an *r-reparametrization* (resp. *strong r-reparametrization*) of F if, for each $\varphi \in S$, $F \circ \varphi$ is of class $\mathcal{C}^{(r)}$ and $(F \circ \varphi)^{(\alpha)}$ is strongly bounded (resp. bounded by 1) for all $\alpha \in \mathbb{N}^{\dim(X)}$ with $|\alpha| \leq r$.

Remark 5.7. If $R = \mathbb{R}$, there is no difference between bounded and strongly bounded.

The following theorem, in the semi-algebraic case, goes back to Yomdin [Yom87b, Yom87a], later generalized by Gromov [Gro87]. In this general form for o-minimal structures it is due to Pila and Wilkie, and is the key input for their counting theorem.

Theorem 5.8 (Reparametrization theorem)**.** *(1) For any $r \in \mathbb{N}$ and any strongly bounded, definable set X, there exists an r-parametrization of X.*

(2) For any $r \in \mathbb{N}$, and any strongly bounded, definable map F, there exists an r-reparametrization of F.

5.2 Determinant method

We prove here Theorem 5.5 on diophantine approximation.

Notation 5.9. For $\alpha = (\alpha_1, \dots, \alpha_n) \in \mathbb{N}^n$ and $x = (x_1, \dots, x_n) \in \mathbb{R}^n$, set $x^\alpha = \prod_{1 \leq i \leq n} x_i^{\alpha_i}$ and $|\alpha| = \alpha_1 + \cdots + \alpha_n$.

Set $L(n, d) := \#\{\alpha \in \mathbb{N}^n \mid |\alpha| = d\}$, $D(n, d) := \#\{\alpha \in \mathbb{N}^n \mid |\alpha| \leq d\}$ and $V(n, d) = \sum_{i=1}^d L(n, i)i$.

We have $L(n, d) = \binom{n+d-1}{d-1}$, $D(n, d) = \sum_{i=0}^d L(n, i) = \binom{n+d}{d}$.

Given $m, n, d \in \mathbb{N}^*$, there is a unique $b = b(m, n, d)$ such that $D(m, b) \leq D(n, d) \leq D(m, b + 1)$. We define

$$B(m, n, d) := V(m, b) + (D(n, d) - D(m, b))(b + 1)$$

and $\varepsilon(m, n, d) = \frac{dD(n,d)}{B(m,n,d)}$.

Using the above explicit formulas, we get, for fixed m, n and as $d \to +\infty$ the asymptotic estimates

$$V(n, d) = \frac{1}{(n + 1)(n - 1)!} d^{n+1}(1 + o(1)),$$

$$b = b(m, n, d) = \left(\frac{m!d^n}{n!}\right)^{\frac{1}{m}} (1 + o(1)),$$

$$B(m, n, d) = \frac{1}{(m + 1)(m - 1)!} \left(\frac{m!}{n!}\right)^{1+\frac{1}{m}} d^{n+\frac{n}{m}}(1 + o(1))$$

and if $m < n$, $\lim_{d \to +\infty} \varepsilon(m, n, d) = 0$.

Lemma 5.10. *Fix $n, d \in \mathbb{N}^*$ and $S \subset \mathbb{R}^n$. Assume that for every finite subset $S_0 \subset S$ of cardinality $D(n,d)$, if one considers the matrix (P^α), whose columns are indexed by multi-indices α, with $|\alpha| \leq d$ and whose rows are indexed by $P \in S_0$, the determinant of (P^α) vanishes. Then there is a non-zero polynomial $f \in \mathbb{R}[X_1, \ldots, X_n]$ of total degree at most d that vanishes at every $P \in S$.*

Proof. Choose S_0 such that the matrix $(P^\alpha)_{P \in S_0, \alpha \in \Delta(n,d)}$ is of maximal rank (among all possible choices of $S_0 \subset S$. Consider $(P^\alpha)_{P \in S_1, \alpha \in A}$, a minor of it with non-vanishing determinant of maximal rank. Choose $\alpha_0 \in \Delta(n,d) \backslash A$ and set $f(x_1, \ldots, x_n) = \det((P^\alpha)_{P \in S_1 \cup \{x\}, \alpha \in A \cup \{\alpha_0\}})$. Then f is a non-zero polynomial of total degree at most d, and from the choice of S_1 and S_0, $f(P) = 0$ for every $P \in S$. $\square$

In order to apply Lemma 5.10 to the image of a map with small derivative, we will have to compute determinants of sums of linear maps. Recall the following facts from linear algebra. If $f : V \to V$ is a linear endomorphism of an n-dimensional vector space V, then the n-th exterior product $\Lambda^n(f)$ is multiplication by the determinant of f. Consider now linear maps $f_1, \ldots, f_r : V \to V$. Then

$$\Lambda^n(f_1 + \cdots + f_r) = \sum_{\sigma = (\sigma_1, \ldots, \sigma_n) \in \{1, \ldots, r\}^n} f_{\sigma_1} \wedge \cdots \wedge f_{\sigma_n}.$$

We see that the term corresponding to σ is non-zero only if $\#\{j \mid \sigma_j = k\} \leq \mathrm{rk}(f_k)$ for every $k = 1, \ldots r$.

Fix a basis $e_1, \ldots, e_n$ of V, and write $f_k(e_i) = \sum_{j=1}^k a_{i,j}^{(k)} e_j$. Then one obtain the corresponding equality of determinants (indexed by i, j)

$$\det(\sum_{k=1}^r a_{i,j}^{(k)}) = \sum_{\sigma = (\sigma_1, \ldots, \sigma_n) \in \{1, \ldots, r\}^n} \det(a_{i,j}^{(\sigma_i)}).$$

Combining this with the observation on ranks, we get

$$\det(\sum_{k=1}^r a_{i,j}^{(k)}) = \sum_{\substack{\sigma \in \{1, \ldots, r\}^n \\ \#\{j \mid \sigma_j = k\} \leq \mathrm{rk}(f_k) \\ \text{for } k = 1, \ldots, r}} \det(a_{i,j}^{(\sigma_i)}).$$

Using these observations, we can prove the key estimate of determinant.

Proposition 5.11. *Let $m, n, d \in \mathbb{N}^*$. Set $\mu = b(m,n,d) + 1$. Then there is a constant $C = C(m,n,d)$ such that the following holds. For every $\varphi :]0,1[^m \subset \mathbb{R}^m \to \mathbb{R}^n$ map of class $\mathcal{C}^{(\mu)}$, with $\left|\varphi^{(\alpha)}(x)\right| \leq 1$ for all $x \in]0,1[^m$ and $\alpha \in \mathbb{N}^n$ with $|\alpha| \leq \mu$, for every $r \in]0,1[$, for every $P_1, \ldots, P_{D(n,d)} \in]0,1[^m$ such that the distance between P_i and P_1 is less than r, we have that the determinant of the matrix $(\varphi(P_i)^\alpha)$ whose row are indexed by $i = 1, \ldots, D(n,d)$ and whose columns are indexed by $\alpha \in \Delta(n,d)$ is less than $K r^{B(m,n,d)}$.*

Proof. Expand each $\varphi(P_i)$ as a Taylor polynomial around P_1 of order $\mu - 1$ and with remainder of order μ. This allows us to view $\varphi(P_i)$ as a polynomial in $P_i - P_1$ of total degree μ and coefficients bounded by 1. Hence for every $\alpha \in \mathbb{N}^n$ with $|\alpha| \leq d$, we can express $\varphi(P_i)^\alpha$ as a polynomial in $P_i - P_1$ with coefficients bounded by d^n. Rewrite the matrix as

$$(\varphi(P_i)^\alpha) = \left(\sum_{k=0}^{\mu} R_{i,\alpha}^{(k)} \right),$$

where for $k < \mu$, each $R_{i,\alpha}^{(k)}$ is an homogeneous polynomial in $P_i - P_1$ of total degree k, and each $R_{i,\alpha}^{(\mu)}$ is a sum of homogeneous polynomials of degree at least μ. For $k < \mu$, the matrix $R^{(k)}$ has rank at most $L(m,k)$, since this is the dimension of the space of homogeneous polynomials of degree k in m variables. From the discussion preceding the Proposition, we get

$$\det(\varphi(P_i)^\alpha) = \sum_{\substack{\sigma \in \{0,\dots,\mu\}^D \\ \#\left\{ j | \sigma_j = k \right\} \leq L(m,k) \\ \text{for } k=1,\dots,r}} \det(R_{i,\alpha}^{(\sigma_i)}).$$

The determinant associated to σ is bounded by a constant (depending only on m,n,d) times $r^{|\sigma|}$. Since $r < 1$, this bound is the biggest when $|\sigma|$ is the smallest, that is when $\#\left\{ j \mid \sigma_j = k \right\} = L(m,k)$. In that case, the exponent is

$$\sum_{i=0}^{\mu-1} L(m,i)i + (D(n,d) - D(m,b))\mu = B(m,n,d).$$

We get the result by summing those bounds. □

We can now prove the main result of this section.

Theorem 5.12. *Let $m,n \in \mathbb{N}$, with $m < n$. There are for each $d \in \mathbb{N}^*$ an integer $\mu = \mu(m,n,d)$ and positive constants $\varepsilon(m,n,d)$ and $C(m,n,d)$ with the following properties. Assume that $\varphi :]0,1[^m \subset \mathbb{R}^m \to \mathbb{R}^n$ is a function of class $\mathcal{C}^{(\mu)}$, with $\left|\varphi^{(\alpha)}(x)\right| \leq 1$ for all $x \in]0,1[^m$ and $\alpha \in \mathbb{N}^m$ with $|\alpha| \leq r$. Set $X = \varphi(]0,1[^m)$. Then for every $H \geq 1$, the set $X(\mathbb{Q}, H)$ is contained in at most*

$$C(k,n,d)H^{\varepsilon(k,n,d)}$$

hypersurfaces of degree at most d. Moreover, $\varepsilon(k,n,d) \to 0$ as $d \to +\infty$.

Proof. Given $H \geq 1$, let C' be the constant given by Proposition 5.11. Fix $\mu = b(m,n,d) + 1$, as in Proposition 5.11. Set $r < H^{\frac{-dD(n,d)}{B(m,n,d)}} C'^{\frac{-1}{B(m,n,d)}}$. Observe that if $P_1, \dots, P_{D(n,d)} \in]0,1[^m$ such that $\varphi(P_i) \in X(\mathbb{Q}, H)$, then there is a positive integer $s \leq H^{dD(n,d)}$ such that $\det(\varphi(P_i)^\alpha) \in \frac{1}{s}\mathbb{Z}$. We further assume that all P_i lies in a box of size r, hence by Proposition 5.11, this determinant is bounded by $C' r^{B(m,n,d)} < H^{-dD(n,d)}$. By comparing those two bounds, we see that this determinant vanishes. We may cover $]0,1[^m$ by $\frac{1}{r^m}$ boxes of size r, or, by choosing r close to $H^{\frac{-dD(n,d)}{B(m,n,d)}} C'^{\frac{-1}{B(m,n,d)}}$, a constant multiple of $H^{\frac{mdD(n,d)}{B(m,n,d)}}$ boxes of radius r. For each such box, we know by Lemma 5.10 that there is a single hypersurface of degree at most d that contains all rational points of height at most H in the image of this box. □

5.3 Yomdin-Gromov parameterizations

Recall Definition 5.6 and the reparametrization theorem, which apply to definable set in an o-minimal ordered field R.

Theorem 5.13 (Reparametrization theorem). *(1) For any $r \in \mathbb{N}$ and any strongly bounded, definable set X, there exists an r-parametrization of X.*

(2) For any $r \in \mathbb{N}$, and any strongly bounded, definable map F, there exists an r-reparametrization of F.

Although having an r-parametrization is apparently not a first order property, we have the following proposition.

Proposition 5.14. *A definable set $X \subset]0,1[^n$ admits a strong r-parametrization if and only if it admits an r-parametrization. A strongly bounded definable map $F : X \subset]0,1[^n \to]0,1[^m$ admits a strong r-reparametrization if and only if it admits an r-reparametrization.*

Proof. Let $c \in \mathbb{N}$ such that there exists an r-parametrization of X, with all derivatives bounded by c. Cover $]0,1[^{\dim(X)}$ with $(2c)^{\dim(X)}$ cubes of size $1/c$, and for such cube C, let $f_C :]0,1[^{\dim(X)} \to C$ be the natural affine bijection. Then precomposing the maps of the r-parametrization of X by f_C for varying C gives the desired strong r-parametrization. □

Before starting the proof of the parametrization theorem, we prove using the compactness theorem that it implies a uniform version.

Corollary 5.15. *If $(X_y)_{y\in Y}$ is a definable family of definable subsets of $]0,1[^n$, and $r \in \mathbb{N}$, then there are finitely many families of definable maps*

$$\left\{\varphi_{i,y} :]0,1[^{k_i} \to]0,1[^n\right\}_{i\in I, y\in Y}$$

such that for each $y \in Y$, for some subset I_0 of I, $\{\varphi_{i,y}\}_{i\in I}$ is a strong r-parametrization of X_y.

Moreover, if $(F_y : X_y \to]0,1[^m)_{y\in Y}$ is a definable family of definable maps, then there are finitely many families of definable maps

$$\left\{\varphi_{i,y} :]0,1[^{k_i} \to]0,1[^n\right\}_{i\in I, y\in Y}$$

such that for each $y \in Y$, for some subset I_0 of I, $\{\varphi_{i,y}\}_{i\in I_0}$ is a strong r-reparametrization of F_y.

Proof. We prove only the first part, the second being similar. We will prove the following weaker result : there is some $N \in \mathbb{N}$ and N families $\{\varphi_{i,c}\}_{c\in C_i}$ of definable maps such that for any $y \in Y$, there is some $I_0 \subset \{1,\dots,N\}$ and some $c_i \in C_i$ such that $\{\varphi_{i,c_i}\}_{i\in I_0}$ is a strong r-parametrization of X_y.

This will imply the full result using definable choice (Proposition 4.24) as follows. Given $y \in Y$ and $I_0 \subset \{1,\dots,N\}$, the condition on $(c_1,\dots,c_N) \in C_1 \times \dots \times C_N$ that $\{\varphi_{i,c_i}\}_{i\in I_0}$ is a strong r-parametrization of X_y is definable. Hence by Proposition

4.24, there is a definable map $\sigma_I = (\sigma_{I,1}, \dots, \sigma_{I,N}) : Y \to C_1 \times \cdots \times C_N$ such that if we can find $c_1, \dots, c_N$ such that $\{\varphi_{i,c_i}\}_{i\in I}$ is a strong r-parametrization of X_y, then $\left\{\varphi_{i,\sigma_{I,i}(y)}\right\}_{i\in I}$ is a strong r-parametrization of X_y. Hence we get the result.

To prove the above statement, we will use the compactness theorem 2.15. Suppose that the statement fails. Let $\mathcal{L}$ be the language of $\mathfrak{R}$, and consider $\mathcal{L}' = \mathcal{L} \cup \{b\}$, where b is a constant tuple of the length of Y. Consider $\mathcal{L}'$-theory $\mathcal{T}'$ consisting of the following sentences:

- $\mathrm{Th}(\mathfrak{R})$: the full $\mathcal{L}$-theory of $\mathfrak{R}$;
- $b \in Y$;
- for every finite sequence of $\mathcal{L}$-formulas $\psi_1(x, y, z), \dots, \psi_N(x, y, z)$, where x is a tuple of k_i variables, y a tuple of m-variables and z a tuple of s_i variables, the assertion that $\forall c_{1,s_1}$, c_{j,s_j}, the sets defined by $\psi_1(x, y, c_{1,s_1}), \dots, \psi_N(x, y, c_{N,s_N})$ do not give a strong r-parametrization of Y_b.

We already seen that such condition is expressible in a first order way. By the compactness theorem, if $\mathcal{T}'$ is finitely satisfiable, then its admits a model $\mathfrak{R}'$. Such a structure is o-minimal by Theorem 4.14, hence by Theorem 5.8, the set X_b (where b is the interpretation of b in the structure $\mathfrak{R}'$) admits a strong r-parametrization. Contradiction with what states $\mathcal{T}'$.

It remains to prove that $\mathcal{T}'$ is finitely satisfiable. Fix a finite subset of it. It contains only finitely many sentences build using formulas $\psi(x, y, z)$. Given our hypothesis, there is a $y \in Y$ such that no subcollection of the (finitely many) functions possibly defined by the finitely many $\psi(x, y, z)$ gives a strong r-parametrization of Y_y. Interpreting b as y, we this that $\mathfrak{R}$ can be seen as model of this finite subset of $\mathcal{T}'$. □

The proof of Theorem 5.8 will consists in proving by induction the following statements:

$B(r)$ If $F :]0,1[\to R$ is strongly bounded, then there exists a r-reparametrization of F such that for each φ in the reparametrization, φ or $F \circ \varphi$ is a polynomial with strongly bounded coefficients.

$R(m,n,r)$ Every strongly bounded, definable map $F : X \subset]0,1[^n \to R^m$ admits an r-reparametrization.

$P(n,r)$ Every strongly bounded, definable set $X \subset R^n$ admits a r-parametrization.

Remark 5.16. Note that $B(r)$ is a strong form of $R(1,1,r)$. We will first prove $B(r)$ by induction. Then we will show the technical fact that for fixed r, $R(1,n,r)$ for every n implies $R(m,n,r)$ for every n and m. The main part of the proof will be that assuming $P(n,k)$ and $R(m,n,k)$ for k up to r, we will prove successively $P(n+1,r)$ and $R(1,n+1,r)$.

Lemma 5.17. *Let $r \geq 2$ and $f :]0,1[\to R$ be a definable map of class $\mathcal{C}^{(r)}$, with $f^{(k)}$ strongly bounded for $0 \leq k \leq r-1$. Assume also that $\left|f^{(r)}\right|$ is decreasing. Define $g :]0,1[\to R$ by $g(x) = f(x^2)$. Then $g^{(k)}$ is strongly bounded for $0 \leq k \leq r$.*

Proof. Using chain rule for derivation, we can write

$$g^{(k)}(x) = \sum_{i=0}^{k} g_{i,k} f^{(i)}(x),$$

where $g_{i,k}$ are polynomials with integer coefficients and such that $g_{k,k}(x) = 2^k x^k$. Since each $g_{i,k}$ is strongly bounded on $]0,1[$, each $f^{(k)}$ is strongly bounded on $]0,1[$ (for $k < r$), and the class of strongly bounded maps form a ring, all we need to show is that $x \mapsto 2^r x^r f^{(r)}(x^2)$ is strongly bounded.

Let $c \in \mathbb{N}$ be a bound for $f^{(r-1)}$. We claim that for every $x \in]0,1[$, $\left|f^{(r)}\right| \leq 4c/x$. This implies the result since we get for every $x \in]0,1[$,

$$2^r x^r f^{(r)}(x^2) \leq 2^r x^r 4c/x^2 = 2^{r+2} c x^{k-2} \leq 2^{r+2} c.$$

To prove the claim, suppose for contradiction that there is some $x_0 \in]0,1[$ such that $\left|f^{(r)}\right| > 4c/x_0$. By Rolle theorem 4.27, there is some $y \in]x_0/2, x_0[$ such that $f^{(r-1)}(x_0) - f^{(r-1)}(x_0/2) = f^{(r)}(y)(x_0 - x_0/2)$. But by hypothesis on $f^{(r)}$, we have $\left|f^{(r)}(y)\right| \geq \left|f^{(r)}(x_0)\right| > 4c/x_0$. Hence

$$2c \geq \left|f^{(r-1)}(x_0) - f^{(r-1)}(x_0/2)\right| > \frac{4c}{x_0}(x_0 - \frac{x_0}{2}) = 2c,$$

contradiction. □

Proposition 5.18. *For every $r \in \mathbb{N}^*$, $B(r)$ holds, that is if $F :]0,1[\to R$, then there exists a r-reparametrization of F such that for each φ in the reparametrization, φ or $F \circ \varphi$ is a polynomial with strongly bounded coefficients.*

Proof. We work by induction on r. For the base case $r = 1$, one use Proposition 4.29 to partition $]0,1[$ intro finitely many intervals $]a,b[$ such that F is of class $\mathcal{C}^{(1)}$ on $]a,b[$, and either $|F'| \leq 1$, or $|F'| > 1$ on $]a,b[$. In the first case $\varphi(x) = a + (b-a)x$ is a parametrization of $]0,1[$, such that $|(F \circ \varphi)'| \leq 1$. In the second case, F is strictly monotone, we set $a' = \lim_{x \to a^+} F(x)$ and $b' = \lim_{x \to b^-} F(x)$. Note that a' and b' are strongly bounded because F is strongly bounded. For $x \in]0,1[$, define $\varphi(x) = F^{-1}(a' + (b' - a')x)$. Then $F \circ \varphi$ is a 1-reparametrization of F on $]a,b[$. We add constant maps to the remaining finitely many points to obtain a 1-reparametrization of F on $]0,1[$.

Fix now $r > 1$ and assume $B(k)$ holds for $k < r$. Let S be an $(r-1)$-reparametrization of F with the extra property. Let $\varphi \in S$ and write $\{\varphi, F \circ \varphi\} = \{g, h\}$ where g is a polynomial with strongly bounded coefficients. Then $g^{(k)}$ exists and is strongly bounded for every k. We know that $h^{(k)}$ exists, is continuous and strongly bounded for $k \leq r-1$. By Corollary 4.30 and the monotonicity theorem, there is a finite partition of $]0,1[$ into points and intervals $]a,b[$ such that on each $]a,b[$, h is of class $\mathcal{C}^{(r)}$ and $\left|h^{(r)}\right|$ is monotonic. We define $\psi :]0,1[\to]a,b[$ by

$$\psi(x) = \begin{cases} a + (b-a)x & \text{if } \left|h^{(r)}\right| \text{ is decreasing} \\ a + (b-a)x & \text{if } \left|h^{(r)}\right| \text{ is increasing.} \end{cases}$$

Then $h \circ \psi :]0,1[\to R$ is of class $\mathcal{C}^{(r)}$, $(h \circ \psi)^{(k)}$ is strongly bounded for $k < r$ and $\left|(h \circ \psi)^{(k)}\right|$ is decreasing. Let ρ be the square map on $]0,1[$. By Lemma 5.17, $h \circ \psi \circ \rho$ has all its derivative up to order r strongly bounded, and $g \circ \psi \circ \rho$ is still a polynomial with strongly bounded coefficients. When $]a,b[$ varies among the finitely many intervals of $]0,1[$, the range of $\varphi \circ \psi \circ \rho$ covers $\mathrm{Im}(\varphi)$ except finitely many points. If we add constants maps, we get the required r-reparametrization of F. □

Lemma 5.19. *Assume that $R(1,n,r)$ holds for every n, r. Then $R(m,n,r)$ holds for every m, n, r.*

Proof. We work by induction on m. It is enough to show that if $n \geq 2$, $F : X \subset R^n \to R^{m-1}$ and $f : X \to R$ are two strongly bounded definable maps that admits an r-reparametrization, then $(F, f) : X \to R^m$ admits an r-reparametrization. To do so, choose S, an r-reparametrization of F and fix $\varphi \in S$. By the induction hypothesis applied to the map $f \circ \varphi$, we get an r-reparametrization S_φ of $f \circ \varphi$. We have $\varphi :]0,1[^k \to X$, with $k \leq n$. Each $\psi \in S_\varphi$ has domain $]0,1[^k$ and by the chain rule for derivation, $(\varphi \circ \varphi)^{(\alpha)}$ is strongly bounded for $|\alpha| \leq r$. Then $\{\psi \in\in S_\varphi \mid \varphi \in S\}$ is an r-reparametrization of (F, f). □

Lemma 5.20. *Assume that $P(k,r)$ and $R(m,k,r)$ for k up to n, then $P(n+1,r)$ holds.*

Proof. Fix $X \subset R^{n+1}$ definable and strongly bounded. We need to show that X admits an r-parametrization. We can assume that X is a cell, and we treat the most difficult case, *i.e* we assume that $X = (f,g)_Y$, with $f, g : Y \subset R^n \to R$ some definable continuous map and Y a cell (of dimension $d \leq n$). Then Y is strongly bounded, so by $P(n,r)$, we can choose S, an r-parametrization of Y. For each $\varphi \in S$, let S_φ be an r-reparametrization of the map $(f,g) : x \in]0,1[^d \mapsto (f(x), g(x)) \in R^2$ (which exists by $R(2,d,r)$). For each $\psi \in S_\varphi$, define $\theta_{\varphi,\psi} :]0,1[^{d+1} \to X$ by

$$\theta_{\varphi,\psi}(x) = (\varphi \circ \psi(x_1, \ldots, x_d), (1 - x_{d+1}) f \circ \varphi \circ \psi(x_1, \ldots, x_d) + x_{d+1} g \circ \varphi \circ \psi(x_1, \ldots, x_d)).$$

The set of $\theta_{\varphi,\psi}$ clearly form an r-parametrization of R. □

To prove the remaining part of the induction, namely the existence of reparametrizations, we will be led to consider definable families of parametrizations (say indexed by $t \in]0,1[$) that we would like to make to converge to a parametrization when $t \to 0$.

Lemma 5.21. *Let $r \geq 1$ and $(F_t)_{t \in]0,1[}$ be a definable family of strongly bounded definable maps $F_t :]0,1[\to R$. Suppose that for every $t \in]0,1[$, the map F_t is of class $\mathcal{C}^{(r)}$, with derivatives up to order r strongly bounded, and that this bound is the same for every $t \in]0,1[$. Set $F_0(x) := \lim_{t \to 0^+} F_t(x)$. Then F_0 is a definable map of class $\mathcal{C}^{(r-1)}$ with derivatives of order up to $r-1$ strongly bounded.*

Proof. Let c be the uniform bound of all derivatives of F_t, for $t \in]0,1[$. Then for fixed $x \in]0,1[$ the definable map $t \in]0,1[\mapsto F_t(x)$ is also bounded by c, hence its limit as $t \to 0^+$ exists and lies in $[-c,c]$. Set $F_0(x) := \lim_{t \to 0^+} F_t(x)$. Then F_0 is strongly bounded. Since $r \geq 1$, it is also continuous (use Rolle's theorem). We now repeat this argument with $k = 1, \ldots, r-1$ to show that F_0 is of class $\mathcal{C}^{(r-1)}$, with strongly bounded k-th derivative. □

Proposition 5.22. *Suppose that $U \subset]0,1[^{n+1}$ is a dense open definable set and $f : U \to R^m$ a definable strongly bounded map such that for each $i \le n$ (but not necessarily $n+1$), $\frac{\partial f}{\partial x_i}$ exists, is continuous and strongly bounded on U. Then for each $r \ge 2$, there is a $(r-1)$-parametrization S of $]0,1[$ and a dense open definable subset $V \subset U$ such that for $\varphi \in S$, if $I_\varphi(x_1,\dots,x_{n+1}) = (x_1,\dots,x_n,\varphi(x_{n+1}))$, then $I_\varphi(V) \subset U$, $f_\varphi = f \circ I_\varphi$ is of class $\mathcal{C}^{(1)}$ on V, and all first derivatives of f_φ are strongly bounded.*

Proof. For simplicity we only prove the result for $m = 1$. The general case can be deduced from it using induction, as in the proof of Lemma 5.19. Fix $r \ge 2$.

By Theorem 4.36, we can find a dense open definable set $W \subset U$ such that f is $\mathcal{C}^1$ on W. For $t, y \in]0,1[$, set

$$W_t(y) := \{x \in]0,1[^n \mid \operatorname{dist}(x, [0,1]^n \times \{y\} \setminus W) \ge t\}.$$

Note that $W_t(y)$ is closed definable, hence if $W_t(y) \ne \emptyset$, the map $x \mapsto \left|\frac{\partial f}{\partial x_{n+1}}(x,y)\right|$ is defined, continuous, and take a maximum value on $W_t(y)$. By definable choice (Proposition 4.24), there is a definable map $s_t(y)$ such that if $W_t(y) \ne \emptyset$, $x \mapsto \left|\frac{\partial f}{\partial x_{n+1}}(x,y)\right|$ attains its maximum on $s_t(y)$. For every $t, y \in]0,1[$ and $x \in W_t(y)$, we have

$$\left|\frac{\partial f}{\partial x_{n+1}}(s_t(y),y)\right| \ge \left|\frac{\partial f}{\partial x_{n+1}}(x,y)\right|.$$

Consider the definable family of definable maps

$$\{g_t : y \in]0,1[\mapsto (s_t(y), f(s_t(y),y)) \in]0,1[^n \times R\}_{t \in]0,1[}$$

(when s_t is undefined, set $s_t(y) = (1/2,\dots,1/2)$). Since f is strongly bounded, each g_t is strongly bounded, with a uniform bound for every $t \in]0,1[$. Since g_t is a one variable definable map, by Proposition 5.18 it admits an r-parametrization S_t. By the compactness theorem, as in the proof of Corollary 5.15, we can assume that the maps in S_t vary in uniform families, for $t \in]0,1[$, and that the strong bound on the derivatives is uniform in $t \in]0,1[$.

Hence by Lemma 5.21, we can consider the limit S_0 of S_t, as $t \to 0$. By splitting the functions in S_0, we can assume that they are all constant or injective, have domain an open interval of $]0,1[$, and that the injective ones have codomain $]0,1[$. Throw away the constant ones, and compose each remaining map with an injective linear function (with coefficients in $[-1,1]$) in order to obtain an $(r-1)$ parametrization S of a cofinite subset of $]0,1[$. (The union of the ranges of maps in S is indeed cofinite in $]0,1[$, otherwise it would miss an open subinterval of $]0,1[$, contradicting the fact that S_t is a parametrization of $]0,1[$ for every $t \in]0,1[$.)

Set $V = (]0,1[^{n+1} \setminus \bigcup_{\varphi \in S} I_\varphi^{-1}(]0,1[^{n+1} \setminus W)) \cap U$. Injectivity and continuity of φ implies that V is a dense open definable set $V \subset U$, and by construction $I_\varphi(V) \subset W$. It remains to show that for $(x_0,y_0) \in V$, $\frac{\partial f_\varphi}{\partial x_i}(x_0,y_0)$ is strongly bounded for $i = 1,\dots,n+1$. By hypothesis, this is clear for $i \le n$ since $(x_0,\varphi(y_0)) \in W$. For $i = n+1$, we reason as follows. Since φ is obtained by precomposing (the restriction of) a map in S_0 by a linear map with bounded coefficients, it is enough to show that for $\psi \in S_0$, and (x_0,y_0) such that $y_0 \in \operatorname{dom}(\psi)$, and $(x_0,\psi(y_0)) \in W$, we have $\psi'(y_0)\frac{\partial f}{\partial x_{n+1}}(x_0,\psi(y_0))$ strongly bounded.

By definition of S_0, there is a $\varphi_t \in S_t$ such that $\lim_{t\to 0^+} \varphi_t(y_0) = \psi(y_0)$ and $\lim_{t\to 0^+} \varphi'_t(y_0) = \psi'(y_0)$. For every small enough $t \in]0,1[$, we have the following :

(1) $x_0 \in W_t(\psi(y_1))$ (since W is open);

(2) $\left|\frac{\partial f}{\partial x_{n+1}}(x_0, \psi(y_0)) - \frac{\partial f}{\partial x_{n+1}}(x_0, \varphi_t(y_0))\right| \leq 1$ and $(x_0, \varphi_t(y_0)) \in W$ (by continuity of $\frac{\partial f}{\partial x_{n+1}}$ on W);

(3) $|\varphi'_t(y_0) - \psi'(y_0)| \leq \left|\frac{\partial f}{\partial x_{n+1}}(x, \psi(y_0))\right|^{-1}$;

(4) $x_0 \in W_t(\varphi_t(y_1))$.

For such a t, we have

$$\begin{aligned}
\left|\psi'(y_0)\frac{\partial f}{\partial x_{n+1}}(x_0,\psi(y_0))\right| &\leq \left|\varphi'_t(y_1)\frac{\partial f}{\partial x_{n+1}}(x_0,\psi(y_0))\right| + 1 \qquad (5.1)\\
&\leq \left|\varphi'_t(y_1)\frac{\partial f}{\partial x_{n+1}}(x_0,\varphi_t(y_0))\right| + |\varphi'_t(y_0)| + 1 \qquad (5.2)\\
&\leq \left|\varphi'_t(y_1)\frac{\partial f}{\partial x_{n+1}}(s_t(\varphi_t(y_0)),\varphi_t(y_0))\right| + |\varphi'_t(y_0)| + 1 \qquad (5.3)
\end{aligned}$$

By construction of S_t, φ'_t is strongly bounded, hence it is enough to show that $\varphi'_t(y_1)\frac{\partial f}{\partial x_{n+1}}(s_t(\varphi_t(y_0)), \varphi_t(y_0))$ is strongly bounded. Since S_t is an r-reparametrization of g_t, we have that $(s_t \circ \varphi_t)'$ and $\frac{d}{dy} f(s_t \circ \varphi_t(y), \varphi_t(y))$ are strongly bounded. That last map is

$$(s_t \circ \varphi_t)'(y_0)\sum_{i=1}^{n}\frac{\partial f}{\partial x_i}((s_t\circ\varphi_t(y_0),\varphi_t(y_0)) + \varphi'_t(y_0)\frac{\partial f}{\partial x_{n+1}}((s_t\circ\varphi_t(y_0),\varphi_t(y_0)).$$

By hypothesis, $\frac{\partial f}{\partial x_i}$ is strongly bounded for $i \leq n$, hence the first term in the above expression is strongly bounded, hence the second one is too. □

Corollary 5.23. *Fix $n, r \geq 1$, U a dense open definable subset of $]0,1[^{n+1}$ and $f : U \to R^m$ a definable strongly bounded map. Assume that for each $\alpha = (\alpha_1, \ldots, \alpha_{n+1}) \in \mathbb{N}^{n+1}$ with $|\alpha| \leq r$ and $\alpha_{n+1} = 0$, $f^{(\alpha)}$ exists, is continuous and strongly bounded on U. Then there exists a dense open definable subset $V \subset U$ and an r-parametrization S of a cofinite subset of $]0,1[$ such that for each $\varphi \in S$, $I_\varphi(V) \subset U$, f_φ is of class $\mathcal{C}^{(r)}$ on V, and all its derivatives $f_\varphi^{(\alpha)}$ for $|\alpha| \leq r$ are strongly bounded on V.*

Proof. Under the hypothesis of the corollary, prove by induction on $k = 0, \ldots, r$ the following statement : there exists a dense open definable subset $V_k \subset U$ and an r-parametrization S_k of a cofinite subset of $]0,1[$ such that for each $\varphi \in S_k$, $I_\varphi(V_k) \subset U$, f_φ is of class $\mathcal{C}^{(r)}$ on V_k, and all its derivatives $f_\varphi^{(\alpha)}$ for $|\alpha| \leq r$ and $\alpha_{n+1} \leq k$ are strongly bounded on V.

Assuming V_k and S_k have been constructed, one applies Proposition 5.22 (with $r+1$) to the maps $f_\varphi^{(\alpha)} : V_k \to R$, for $\varphi \in S_k$ and $\alpha \in \mathbb{N}^{n+1}$ with $|\alpha| \leq r$ and $\alpha_{n+1} \leq k$. One obtains a parametrization $S$of a cofinite subset of $]0,1[$ and a dense definable open subset $V_{k+1} \subset V_k$ and one can check that the collection $S_{k+1} := \{\varphi \circ \psi \mid \varphi \in S_k, \psi \in S\}$ has the required properties. □

Proof of the parametrization theorem. We work by induction on n. By Proposition 5.18, Lemma 5.19 and Lemma 5.20, it only remains to show that assuming $P(k,r)$ and $R(m,k,r)$ for k up to n, and $P(n+1,r)$ holds, then $R(1,n+1,r)$ holds. That is, we need to construct an r-reparametization of a definable strongly bounded $F:]0,1[^{n+1}\to R$.

Since F is strongly bounded, up to cutting the image into finitely definable subsets contained into intervals of length 1, we can assume that the image of F is $]0,1[$.

By $R(1,n,r)$, for each $t\in]0,1[$, there is a strong r-reparametrization S_t of the map $F_t:]0,1[^n\to]0,1[: x\mapsto f(x,t)$.

By Corollary 5.15, we can assume that the strong r-reparametrizations S_t come in a definable families $(S_t)_{]0,1[}=\{\varphi_{1,t},\ldots,\varphi_{N,t}\}_{t\in]0,1[}$. For each $i=1,\ldots,N$, set $F_i(x,t)\in]0,1[^{n+1}\mapsto F(\varphi_{i,t}(x),t)$. Set

$$\bar{F}=(\varphi_1,\ldots,\varphi_N,F_1,\ldots,F_N):]0,1[^{n+1}\to R^{nN+N}.$$

Since S_t is an r-reparametrization of F_t for $t\in]0,1[$, the hypothesis of Corollary 5.23 are satisfied by $\bar{F}$. Hence we get a dense open definable set $V\subset]0,1[^{n+1}$, and an r-parametrization S with the required properties. If we had $V=]0,1[^{n+1}$ and S a parametrization of the full $]0,1[$, then the collection of maps φ_{i_ψ}, for $\psi\in S$ and $i=1,\ldots,N$ would be the required reparametrization. In general, the images (of $]0,1[^{n+1}$) of those maps cover $]0,1[^{n+1}$ but finitely many hyperplanes of equations $x_{n+1}=a$. Hence if one restrict them to the open dense subset V, their images is a subset of $]0,1[^{n+1}$ of codimension k, for some $k\le n$ (by Proposition 4.20). Denote by $Y\subset]0,1[^{n+1}$ such a complement of dimension k.

Using $P(n+1,r)$, we can choose an r-parametrization S_1 of V, and an r-parametrization S_2 of the k-dimensional set Y. Using $R(1,k,r)$, for each $\chi\in S_2$, we can find an r-reparametrization S_χ of $F\circ\chi:]0,1[^k\to]0,1[$.

The required r-parametrization of F is now given by

$$\left\{\varphi_{i_\psi}\circ\chi\mid i=1,\ldots,N,\psi\in S,\chi\in S_1\right\}\cup\{\chi\circ\theta\mid\chi\in S_2,\theta\in S_\chi\},$$

where the domain of $\theta\in S_\chi$ is trivially extended from $]0,1[^k$ to $]0,1[^{n+1}$. □

5.4 Proof of the counting theorem

We now work in an o-minimal structure over the field of real numbers $\mathbb{R}$.

Proposition 5.24. *Let $X=(X_y)_{y\in Y}$ be a definable family of subsets $X_y\subset]0,1[^n$ of dimension $k<n$. Let $\varepsilon>0$. There are $d=d(\varepsilon,k,n)$ and $C=C(Z,\varepsilon)$ such that for every $y\in Y$, and $H\ge1$, the set $X_y(\mathbb{Q},H)$ is contained in the union of at most CH^ε hypersurfaces of degree at most d.*

Proof. Apply Theorem 5.5 to a strong r-parametrization of $(X_y)_{y\in Y}$ given by Corollary 5.15 of the parametrization theorem, where r is the $\mu(m,n,d)$ appearing in the statement of Theorem 5.5 and d is chosen large enough such that $\varepsilon(k,n,d)\le\varepsilon$. □

Exercise 5.25. Prove directly the counting theorem for $X\subset\mathbb{R}^2$ a definable curve (a set of dimension 1).

If $X \subset \mathbb{R}^n$ is definable and $k \leq n$, we denote by $\mathrm{reg}_k(X)$ the set of $\mathcal{C}^{(1)}$-smooth points of X of dimension k, that is, a point $x \in \mathrm{reg}_k(X)$ if $x \in X$ and there is a box B around x such that $X \cap B$ is a $\mathcal{C}^{(1)}$ differential subvariety of $\mathbb{R}^n$ of dimension k. Note that $\mathrm{reg}_k(X)$ is definable (since charts can be given by coordinate projections, and being a $\mathcal{C}^{(1)}$-differomophism is a definable condition). Observe also (using the cell decomposition theorem) that if X is of dimension k, then $\mathrm{reg}_k(X)$ is a dense subset of X, hence $X \backslash \mathrm{reg}_k(X)$ is of dimension less than k.

Recall the statement of the counting theorem in its general version.

Theorem 5.26 (Pila-Wilkie counting theorem, general version)**.** *Let $X \subset \mathbb{R}^{n+m}$ be a set definable in an o-minimal expansion of $(\mathbb{R}, <, +, -, \cdot)$. Then for every $\varepsilon > 0$, there is a constant $C = C(X, \varepsilon)$ and a definable set $W = W(X, \varepsilon) \subset X$ with the following properties. For every $y \in \mathbb{R}^m$, $W_y \subset (X_y)^{\mathrm{alg}}$ and for every $H \geq 1$,*

$$\#(X_y \backslash W_y)(\mathbb{Q}, H) \leq C H^{\varepsilon}.$$

Proof. Points of height at most H are stable under maps $x \mapsto \pm x^{\pm x}$, hence we can assume $X \subset [0,1]^n \times \mathbb{R}^m$. By induction on n, we can further assume $X \subset]0,1[^n \times \mathbb{R}^m$.

Note that if $X = A \cup B$, then if $y \in Y$, then $A_y^{\mathrm{alg}} \cup B_y^{\mathrm{alg}} \subset X_y^{\mathrm{alg}}$, hence if the theorem holds for A and B it also holds for X.

We will prove the theorem by induction on the fiber dimension of X (*i.e;* dimension of the fibers X_y, for $y \in Y$). If the fiber dimension is 0, then by uniform finiteness (UF) from Theorem 4.12, there is a uniform bound c for the number of points in every fiber Y_y, hence the theorem holds with $C(X, \varepsilon) = c$.

Suppose now that $k > 0$ and that the theorem holds for all families $X = (X_y)_{y \in Y} \subset]0,1[^{n'} \times \mathbb{R}^{m'}$ such that the fiber dimension is at most $k-1$. Let $X = (X_y)_{y \in Y} \subset]0,1[^n \times \mathbb{R}^m$ be definable with fiber dimension k. Suppose that $k = n$. If $x \in \mathrm{reg}_n(X_y)$, then X_y contains a box around x hence $x \in X_y^{\mathrm{alg}}$. As observed above, the set $R_k(X) = \{(x,y) \in X \mid x \in \mathrm{reg}_k(X_y)\}$ is definable and the fibers of $R_n(X)$ are contained in the algebraic part of the fibers of X. Hence the conclusion of the theorem for $R_n(X)$ holds with $W(R_n(X), \varepsilon) = R_n(X)$. By dimension theory, the fiber dimension of $X \backslash R_n(X)$ is at most $k-1$, hence the theorem holds for $X \backslash R_n(X)$ by induction, hence for X as well.

We now assume that $k < n$. Let S be the set of coordinates projections $\pi : \mathbb{R}^n \to \mathbb{R}^{k+1}$ and let $q = \#S$. By Proposition 5.24 applied to $\pi(X)$, for $\pi \in S$, there are d and $\alpha(X, \varepsilon)$ such that for every $y \in Y$, every $\pi \in S$ and $H \geq 1$, $\pi(X_y)(\mathbb{Q}, H)$ is contained in at most $\alpha(X, \varepsilon) H^{\frac{\varepsilon}{2q}}$ hypersurfaces of degree at most d. Hence $X_y(\mathbb{Q}, H)$ is contained in at most $\alpha(X, \varepsilon)^q H^{\frac{\varepsilon}{2}}$ cylinders on hypersurfaces of degree at most d in a subspace of dimension $k+1$.

Let $T \subset \mathbb{R}^p$ be a subset parametrizing hypersurfaces of degree d in $\mathbb{R}^{k+1}$. Note that T is in bijection with $\mathbb{P}^{\nu}(\mathbb{R})$ with $\nu = \binom{k+1+d}{d} - 1$, hence we can assume $T \subset [0,1]^p$. Then each

$$t = (t_\pi)_{\pi \in S} \in \prod_{\pi \in S} T$$

correspond to a choice of an hypersurface $L = L(t_\pi)$ of degree at most d in each

$k+1$-dimensional coordinate subspace of $\mathbb{R}^n$. Consider the definable family

$$\Sigma = \left\{ (x,y,t) \mid \pi(x) \in L(t_\pi), \pi \in S, t \in \prod_{\pi \in S} T \right\} \subset \mathbb{R}^n \times \mathbb{R}^m \times \mathbb{R}^{pq}.$$

Note that every fiber $\Sigma_{y,t}$ is a closed algebraic set in $\mathbb{R}^n$ of dimension at most k.

Replace X by

$$\left\{ (x,y,t) \mid (x,y) \in X, t \in \prod_{\pi \in S} T \right\} \subset \mathbb{R}^n \times \mathbb{R}^m \times \mathbb{R}^{pq}.$$

Note that the fiber of this set over (y,t) is X_y, hence proving the theorem for this new set is the same as proving it for X.

The fiber dimension of $X \cap \Sigma$ is at most k. Set

$$A_1 = \{(x,y,t) \in X \cap \Sigma \mid x \notin \mathrm{reg}_k(X \cap \Sigma_{y,t})\},$$

$$A_2 = \{(x,y,t) \in X \cap \Sigma \mid x \notin \mathrm{reg}_k(X_{y,t})\},$$
$$A_3 = \{(x,y,t) \in X \cap \Sigma \mid x \notin \mathrm{reg}_k(\Sigma_{y,t})\},$$

For $i = 1,2,3$, the fiber dimension of A_i is at most $k-1$, hence by induction one has at most $c(A_i, \frac{\varepsilon}{2})H^{\frac{\varepsilon}{2}}$ on $A_i \backslash W(A_i, \frac{\varepsilon}{2})(\mathbb{Q}, H)$, where $W(A_i, \frac{\varepsilon}{2})$ is contained in the algebraic part of A_i.

Let B the set of points in $X \cap \Sigma$ that are regular of dimension k in their fibers in X, Σ and $X \cap \Sigma$.

Let $(x,u) \in B$. There is a small enough box $D \subset \mathbb{R}^n$ around x such that $D \cap X_u$, $D \cap \Sigma_u$ and $D \cap (X \cap \Sigma)_u$ are all $\mathcal{C}^{(1)}$-manifolds of dimension k. Since $(X \cap \Sigma)_u$ is included in both X_u and Σ_u, those three manifolds coincide if D is small enough. Since Σ is semi-algebraic, $D \cap \Sigma_u =$ is a semi-algebraic set of dimension $k \geq 1$, hence $x \in B_u^{\mathrm{alg}} \subset X_u^{\mathrm{alg}}$. Hence the theorem holds for B with $W(B, \varepsilon) = B$.

Now let $y \in Y$ and $H \geq 1$. Let $x \in X_y(\mathbb{Q}, H)$. Hence for $\pi \in S$, $\pi(x) \in \pi(X_{y,t})(\mathbb{Q}, H)$ hence lies in one of the hypersurfaces t_π. Hence x lies in at most $\alpha(X, \varepsilon)^q H^{\frac{\varepsilon}{2}}$ fibers of $X \cap \Sigma$.

Moreover, working in one of those fibers, either x lies in A_i for $i = 1, 2$ or 3, in which case the number of such x outside $W(A_i, \frac{\varepsilon}{2})$ is bounded by $c(A_i, \frac{\varepsilon}{2})H^{\frac{\varepsilon}{2}}$. Otherwise x lies in B. This concludes the proof. □

6 The Pila-Zannier strategy

We expose in this section the Pila-Zannier strategy. We will focus on the proof of the Manin-Mumford conjecture using this strategy, and only sketch its application to the André-Oort conjecture for products of modular curves in the last section.

6.1 Abelian varieties

We introduce here abelian varieties, skipping all but the most elementary proofs. We refer to Milne [Mil08] or Mumford [Mum08] (in the algebraically closed case) for complete proofs.

An abelian variety is defined as a higher dimensional generalization of elliptic curves.

Definition 6.1. An *elliptic curve* over a field k is an object satisfying one of the following equivalent definitions:

(1) ($\mathrm{char}(k) \neq 2, 3$): a projective plane curve with equation of the form

$$ZY^2 = X^3 + aXZ + bZ^3, \text{ where } 4a^3 + 27b^2 \neq 0;$$

(2) a non-singular projective curve of genus one together with a distinguished point;

(3) a non-singular pojective curve together with a group structure defined by regular maps;

(4) ($k = \mathbb{C}$) an algebraic curve E such that $E(\mathbb{C}) \simeq \mathbb{C}/\Lambda$ (complex analytic isomorphism), for some lattice $\Lambda \subset \mathbb{C}$.

Sketch of the equivalences.

(1) $\to$ (2) The condition $4a^3 + 27b^2 \neq 0$ implies that E is non-singular. Since it is defined by an equation of degree 3, it is of genus 1.

(2) $\to$ (1) Recall the Riemann-Roch theorem, that states that for a divisor D on a smooth projective curve C and K_C a canonical divisor if $L(D) = \left\{f \in \bar{k}(X) \mid \mathrm{Div}(f) + D \geq 0\right\}$, then

$$\dim_k(L(D)) - \dim_k(L(K_C - D) = \deg(D) + 1 - \mathrm{genus}(C).$$

In particular, if $\deg(D) > 2g - 2$, then

$$\dim_k(L(D)) = \deg(D) + 1 - \mathrm{genus}(C).$$

Apply the theorem successively to 2∞ and 3∞ to find $x, y \in k(D)$ with exactly one pole at ∞ of order 2 and 3. Use x and y to construct an embedding

$$p : P \in E \backslash \{\infty\} \mapsto [x(P) : y(P) : 1] \in \mathbb{P}^2_k$$

Use again Riemann-Roch (with 6∞) to find that x and y satisfy an equation of the form (1) (use the fact that $L(6\infty)$ contains the 7 functions $1, x, y, x^2, y^2, xy, x^3$), which shows that the image of E by p is a curve of the form (1).

(1) $\to$ (3) Let $\mathrm{Div}^0(E)$ be the group of degree zero divisors on E, and $\mathrm{Pic}^0(E)$ its quotient by principal ideals. Riemann-Roch theorem shows that

$$P \in E(k) \mapsto [P] - [\infty] \in \mathrm{Pic}^0(E)$$

is a bijection, hence it induces a group law structure on $E(k)$. One can check that it coincide with the one defined using chords and tangents, hence the addition and inverse map are regular.

(3) $\to$ (2) If $k \subset \mathbb{C}$, for $a \in E(\mathbb{C})$, let t_a the translation $x \mapsto x + a$. If $a \neq 0$, by Lefschetz fixed point formula,

$$0 = \mathrm{tr}(t_a) = \sum_{i=0}^{2} (-1)^i \mathrm{tr}(t_a \mid \mathrm{H}^i_{\mathrm{Betti}}(E(\mathbb{C}), \mathbb{Q}).$$

By continuity of $a \in E(\mathbb{C}) \mapsto \mathrm{tr}_a \in \mathbb{Z}$, the Euler characteristic of E is $\chi(E) = \mathrm{tr}(t_0) = 0$, *i.e.* E is of genus 0. For general k, one can use the same argument with étale cohomology groups and the Grothendieck-Lefschetz fixed point formula instead.

(2) $\to$ (4) A Riemann surface of genus one is a torus, *i.e* of the form $\mathbb{C}/\Lambda$ for some lattice Λ.

(4) $\to$ (1) The Weierstrass function $\wp : \mathbb{C}/\Lambda \to \mathbb{C}$ is defined by

$$\wp(x) = \frac{1}{z^2} + \sum_{\lambda \in \Lambda \setminus \{0\}} \left(\frac{1}{(x+\lambda)^2} - \frac{1}{\lambda^2} \right).$$

One can check that $x \in \mathbb{C}/\Lambda \setminus \{0\} \mapsto [\wp(x) : \wp'(x) : 1] \in \mathbb{P}^2_{\mathbb{C}}(\mathbb{C})$ defines the desired embedding. $\square$

An abelian variety is defined by a generalization of condition (3).

Definition 6.2. An *abelian variety* over a field k is a proper connected group variety defined over k, *i.e*, a proper variety A over k, together with regular maps $m : A \times_k A \to A$, $i : A \to A$ defined over k and $e \in A(k)$ that satisfies the axioms of a group.

An abelian subvariety B of A is a subvariety that is also a subgroup.

Exercise 6.3. The group law of an abelian variety is commutative. Hint: use the fact that if X and Y are proper connected and $p : X \times_k Y \to U$ a regular map such that there are $u \in U(k), x \in X(k), y \in Y(k)$ such that $p(X \times \{y\}) = p(\{x\} \times Y) = \{u\}$, then $p(X \times Y) = \{u\}$.

Theorem 6.4. *An abelian variety is projective.*

Recall that a lattice $\Lambda \subset \mathbb{C}^g$ is a discrete subgroup such that $\mathbb{C}^g/\Lambda$ is compact (*i.e.* the $\mathbb{R}$-vector space spanned by Λ is $\mathbb{C}^g$).

Theorem 6.5. *If A is an abelian variety over $k \subset \mathbb{C}$ of dimension g, there is a lattice $\Lambda \subset \mathbb{C}^g$ such that $\mathbb{C}^g/\Lambda \simeq A(\mathbb{C})$, the isomorphism being an isomorphism of complex Lie groups.*

We call the (analytic) projection map $\pi : \mathbb{C}^g \to A(\mathbb{C})$ the exponential (or uniformization) map. Hence the set complex points of an abelian variety defined over a subfield of $\mathbb{C}$ is a complex tori.

Remark 6.6. Contrary to the case of elliptic curves, if $g > 1$, it is not true that for every lattice $\Lambda \subset \mathbb{C}^g$, the torus $\mathbb{C}^g/\Lambda$ admits a structure of abelian variety. The lattice Λ must admit a *Riemann form* for this to be true.

Proposition 6.7. *Let A be an abelian variety defined over a field k of characteristic zero, $g = \dim(A)$. For every $N \geq 1$, the set N-torsion points of $A(k^{\mathrm{alg}})$ is in bijection with*

$$A(k^{\mathrm{alg}})[n] \simeq (\mathbb{Z}/N\mathbb{Z})^{2g}.$$

Proof when $k \subset \mathbb{C}$. Since $A(\mathbb{C}) \simeq \mathbb{C}^g/\Lambda$, we have

$$A(\mathbb{C})[N] \simeq \mathbb{C}^g/\Lambda[N] = \frac{1}{N}\Lambda/\Lambda = \frac{1}{N}\mathbb{Z}^{2g}/\mathbb{Z}^{2g} = (\mathbb{Z}/N\mathbb{Z})^{2g}.$$

□

The following result of Masser [Mas84] will be crutial for the proof of the Manin-Mumford conjecture.

Theorem 6.8 (Masser)**.** *Let A be an abelian variety defined over a number field k and of dimension g. There exists constants $c > 0$ depending on A and k and $\rho > 0$ depending only on g such that for all torsion points $P \in A(\bar{k})$ of order exactly N,*

$$[k(P) : k] \geq cN^{\rho}.$$

6.2 Manin-Mumford conjecture

The Manin-Mumford conjecture, first proved by Raynaud [Ray83], is the following statement.

Theorem 6.9. *Let A be an abelian variety defined over $\mathbb{C}$. Let $V \subset A$ be an irreducible subvariety. If $V(\mathbb{C}) \cap A^{\mathrm{tor}}$ is Zariski-dense in V, then V is a translate of an abelian subvariety of A by a torsion point.*

We will prove this theorem in the case where A, V are defined over a number field k, following the strategy of Pila and Zannier [PZ08].

We in fact prove the seemly weaker statement that if V does not contains any translate of a non-trivial abelian suvariety of A, then $V(\mathbb{C}) \cap A^{\mathrm{tor}}$ is finite.

Fix a subvariety V of A, and assume that V does not contain any translate of a (non-trivial) abelian subvariety of A. We need to show that $V(\mathbb{C}) \cap A^{\mathrm{tor}}$ is finite. Since A is an abelian variety, say of dimension g, by Theorem 6.5 there is a lattice $\Lambda \subseteq \mathbb{C}^g$ such that $\mathbb{C}^g/\Lambda$ is complex-analytically isomorphic to $A(\mathbb{C})$, and the isomorphism respect the group structures. Via this identification, the projection map $\pi : \mathbb{C}^g \to A(\mathbb{C})$ is then complex analytic. Use a basis of Λ to identify $\mathbb{C}^g$ with $\mathbb{R}^{2g}$. With this identification, we have $\Lambda = \mathbb{Z}^g$. Set $F = [0, 1[^{2g}$. F is a bounded fundamental domain of $\mathbb{C}^g$ under the action of Λ. Since $\pi : \mathbb{C}^g \to A(\mathbb{C})$ is complex analytic and F is bounded, one can cover $\bar{F}$ by finitely many complex boxes such that on each box, π is given by a tuple of converging complex power-series. Separating real and imaginary parts, one sees that one each box, π is definable in $\mathbb{R}_{\mathrm{an}}$, hence the restriction of π to F is definable in $\mathbb{R}_{\mathrm{an}}$. Set $W = \pi_{|F}^{-1}(V(\mathbb{C}))$.

We will use the following theorem, that will be discussed and proved later.

Theorem 6.10 (Ax-Lindemann-Weierstrass). *Let A be an abelian variety of dimension g over $\mathbb{C}$ and $\pi : \mathbb{C}^g \to A(\mathbb{C})$ the exponential map. Let V be a complex algebraic subvariety of A and Y a maximal irreducible complex algebraic subvariety contained in $\pi^{-1}(V)$. Then $\pi(Y)$ is a translate of an abelian subvariety of A.*

To relate the conclusion of the Ax-Lindemann-Weierstrass, we use the following lemma, which proof is also postponed below.

Lemma 6.11. *Let Z a complex analytic set in $\mathbb{C}^g$ and $X \subset Z$ a connected irreducible real semi-algebraic subset of Z. Then there is a complex algebraic variety X' such that $X \subset X' \subset Z$.*

By irreducible semi-algebraic set, we mean that the Zariski-closure of X is irreducible, *i.e.* X cannot be written as a union $X = X_1 \cup X_2$ of two proper semi-algebraic subsets which are closed in the restriction to X of the Zariski topology on $\mathbb{R}^{2g}$

Introduce the special locus $\mathrm{SpL}(V)(\mathbb{C}) =$

$$\{x \in V(\mathbb{C}) \mid x + B \subset V(\mathbb{C}), \text{ for some abelian subvariety } B \text{ with } \dim(B) > 0\}.$$

Our assumption that V does not contain any translate of a non-trivial abelian subvariety of A implies that $\mathrm{SpL}(V)(\mathbb{C}) = \emptyset$. For the proof of the general case, it is however better to "forget" that we know this.

Note that the theorem and the lemma imply that $W^{\mathrm{alg}} = \pi_{|F}^{-1}(\mathrm{SpL}(V)(\mathbb{C}))$. Indeed, the reverse inclusion is clear. Conversely, if $x \in W^{\mathrm{alg}}$, then it is contained an infinite irreducible semi-algebraic set W, contained in the analytic set $\pi^{-1}(V)$. By Lemma 6.11, there is a (maximal) complex algebraic variety X' such that $X \subset X' \subset \pi^{-1}(V)$ and X' is of positive dimension since X is infinite. By Theorem 6.10, $\pi(X')$ is a translate of an abelian subvariety of A of positive dimension, hence $\pi(x) \in \mathrm{SpL}(V)(\mathbb{C})$.

We need to show that $V \backslash \mathrm{SpL}(V)(\mathbb{C})$ contains only finitely many torsion points.

The torsion points on A correspond via π to points in $\mathbb{Q}^{2g}$, hence to rational points in W. One then needs to show that $W \backslash W^{\mathrm{alg}}$ contains only finitely many rational points. By the Pila-Wilkie theorem, for every $\varepsilon > 0$ there is a constant C such that for any $H \geq 1$,

$$\#W(\mathbb{Q}, H) \leq CH^{\varepsilon}.$$

On the other hand, if $x \in W$ is a rational point of height exactly H, then $\pi(x)$ is a torsion point of A of order H. Recall that A is defined over a number field K. By Masser's theorem 6.8, there exist constants $c > 0$ and $\rho > 0$ such that for any torsion point P of A of order H, we have

$$[K(P) : K] \geq cH^{\rho}.$$

For any such P, all the Galois conjugates of P are also torsion points, providing at least cH^{ρ} different rational points in W. Comparing with the bound provided by the Pila-Wilkie theorem (say for $\varepsilon = \rho/2$), we see that all the rational points in $W \backslash W^{\mathrm{alg}}$ are of height bounded by some H_0, hence there are only finitely many of them.

It remains to prove Lemma 6.11.

Proof of Lemma 6.11. Let S be the Zariski closure of X in $\mathbb{R}^{2g}$. By irreducibility of X, any cell of X of maximal dimension must be Zariski dense in X. Hence S is geometrically irreducible, *i.e* S is irreducible over $\mathbb{C}$ and $\dim(S) = \dim(X)$.

Set $f : (x_1, y_1, \ldots, x_g, y_g) \in \mathbb{C}^{2g} \to (x_1 + iy_1, \ldots, x_g + iy_g) \in \mathbb{C}^g$ and let ι be the inclusion of $\mathbb{R}^{2g}$ in $\mathbb{C}^{2g}$. The composition $f \circ \iota$ is the isomorphism $\mathbb{R}^{2g} \to \mathbb{C}^g$. Let S_1 be the closure of $\iota(S)$ in the complex Zariski topology on $\mathbb{C}^{2g}$ and $S_2 = f(S_1)$. By Chevalley's theorem (see Example 2.26), S_2 is a constructible set. Hence the closure of S_2 in the Euclidean topology is a complex algebraic set. We denote it by X' and claim that it has the required property $X \subset X' \subset Z$. The first inclusion is clear, so it remains to prove $X' \subset Z$.

Claim: There is no proper closed analytic set of S_1 containing $\iota(X)$.

Assuming the claim, we apply it to $f^{-1}(Z) \cap S_1$, which contains $\iota(X)$ by hypothesis, hence find that $f^{-1}(Z) \cap S_1 = S_1$, so $f(S_1) \subset Z$. Since Z is closed and $f(S_1)$ is dense in X' for the Euclidean topology, we conclude that $X' \subset Z$.

It remains to prove the claim. Fix U, the smallest analytic subset of S_1 containing $\iota(X)$. Let $x \in X$ be a point in a cell of maximum dimension of X such that U is smooth at $x_1 = \iota(x)$. Note that such a point exists by minimality of U: the set of smooth points of U is a non-empty open subset of U, hence by minimality $\iota(X)$ is not included in U^{sing}. Hence the set of $x \in X$ such that $\iota(X)$ is smooth in U is an non-empty open subset of X, hence intersect a cell of maximum dimension of X. Since x is in a cell of maximum dimension of X, locally around x, X and S coincide. Since $\iota(X) \subset U$, we deduce that $T_{x_1}\iota(S) \subset T_{x_1}U$. In the above inclusion $T_{x_1}\iota(S)$ is a real vector space and $T_{x_1}U$ is a complex vector space, hence $T_{x_1}U$ contains the complex vector space spanned by $T_{x_1}\iota(S)$, which is $T_{x_1}S_1$. Since U is smooth at x_1, $\dim_{x_1}(U) = \dim(T_{x_1}U)$. Hence we have

$$\dim(T_{x_1}S_1) \leq \dim(T_{x_1}U) = \dim_{x_1}(U) \leq \dim_{x_1}(S_1) \leq \dim(T_{x_1}S_1),$$

so $\dim_{x_1}(U) = \dim_{x_1}(S_1)$. Since $U \subset S_1$ and S_1 is irreducible, we conclude that $U = S_1$. □

Let us explain finally how to prove the full statement of the Manin-Mumford conjecture, namely that if A, V are defined over a number field, with V algebraic irreducible such that $V(\mathbb{C}) \cap A^{\text{tor}}$ is Zariski dense in V, then $V = b + B$ where B is an abelian subvariety of A and b a torsion point.

We explain first why we can assume that the stabilizer of V in A is trivial. If $S = \{a \in A \mid a + X \subset X\}$ is positive dimensional, then the quotient $q : A \to A/S$ is an abelian variety, the image $\bar{V} = q(V)$ is an irreducible subvariety of A/S, with dense subset of torsion points. Then if we know that $\bar{V}$ is a translate of an abelian subvariety B' by a torsion point in A/S, say $bvarV = q(b) + B'$, then the set $b + S \subset A$ contains a torsion point, hence $V = b + S + B'$ and contains a torsion point, hence we can assume b is itself a torsion point.

So we assume for now on that the stabilizer of V in A is trivial. We will show our assumption that set of torsion points in V is dense in V implies that V is of dimension zero, *i.e.* V is a torsion point, which is what we needed to show. Introduce the special locus $\mathrm{SpL}(V)(\mathbb{C}) =$

$$\{x \in V(\mathbb{C}) \mid x + B \subset V(\mathbb{C}), \text{ for some abelian subvariety } B \text{ with } \dim(B) > 0\}.$$

One can show that $\mathrm{SpL}(V)$ is in fact an algebraic subvariety of V, and that, assuming $\dim(V) > 0$, $\mathrm{SpL}(V) = V$ if and only if the stabilizer of V is positive dimensional. Since the later is trivial, all we need to show is that the set of torsion points in $V \backslash \mathrm{SpL}(V)$ is finite.

But this is precisely what we did in the proof above using the Pila-Wilkie theorem.

6.3 Functional transcendence and the Ax-Lindemann-Weierstrass theorem

We prove here the Ax-Lindemann-Weierstrass theorem for abelian varieties, using the Pila-Wilkie counting theorem and ideas of Pila, Ullmo and Yafaev. We give first a bit of context on functional transcendence.

Recall the (wide open) Schanuel conjecture, which implies every transcendence result between complex numbers involving the exponential function that one can dream of.

Conjecture 6.12 (Schanuel). *Let $x_1, \ldots, x_n \in \mathbb{C}$ be linearly independent over $\mathbb{Q}$. Then*

$$\mathrm{tr.d.}_{\mathbb{Q}}(x_1, \ldots, x_n, \exp(x_1), \ldots, \exp(x_n)) \geq n.$$

It implies in particular the Lindemann-Weierstrass theorem:

Theorem 6.13 (Lindemann-Weierstrass). *Let $x_1, \ldots, x_n$ be algebraic numbers that are linearly independent over $\mathbb{Q}$. Then $\exp(x_1), \ldots, \exp(x_n)$ are algebraically independent over $\mathbb{Q}$.*

Using differential-algebra, Ax has established in [Ax71] functional analogs of these statements.

Recall that for $z_1, \ldots, z_n \in K$, where K is a field containing $\mathbb{C}$, we say that $z_1, \ldots, z_n$ are *linearly independent over $\mathbb{Q}$ modulo $\mathbb{C}$* (abbreviated by l.i/$\mathbb{Q}$ mod $\mathbb{C}$) if there are no non-trivial relation of the form

$$q_1 z_1 + \cdots + q_n z_n = c,$$

where $q_1, \ldots, q_n \in \mathbb{Q}$ and $c \in \mathbb{C}$. We say that they are algebraically independent over $\mathbb{C}$ if there a no non-trivial polynomial $P(X_1, \ldots, X_n)$ with complex coefficients such that $P(z_1, \ldots, z_n) = 0$. The transcendence degree of $\mathrm{tr.d.}_{\mathbb{C}}(z_1, \ldots, z_n)$ is the biggest k such that a subset of size k of $z_1, \ldots, z_n$ is algebraically independent over $\mathbb{C}$.

We consider $\pi : \mathbb{C}^n \to (\mathbb{C}^*)^n$ given by $\pi(z_1, \ldots, z_n) = (\exp(z_1), \ldots, \exp(z_n))$. Let $W \subset U \subset \mathbb{C}^n$ a complex analytic variety of some open subset U of $\mathbb{C}^n$. The coordinate functions $z_1, \ldots, z_n$ and $\exp(z_1), \ldots, \exp(z_n)$ are meromorphic on W.

Theorem 6.14 (Ax-Schanuel). *In the above situation, if the $z_1, \ldots, z_n$ (viewed as meromorphic functions on W) are linearly independent over $\mathbb{Q}$ modulo $\mathbb{C}$, then*

$$\mathrm{tr.d.}_{\mathbb{C}}(z_1, \ldots, z_n, \exp(z_1), \ldots, \exp(z_n)) \geq n + \dim(W).$$

Exercise 6.15. Show that Ax-Schanuel theorem implies that if $f_1, \ldots, f_k$ are meromorphic functions one some open subset $U \subset \mathbb{C}^n$ such that $f_1, \ldots, f_k$ are l.i./$\mathbb{Q}$ mod $\mathbb{C}$, then

$$\mathrm{tr.d.}_{\mathbb{C}}(f_1, \ldots, f_k, \exp(f_1), \ldots, \exp(f_k)) \geq k.$$

Hint: show first the result assuming that $z_1, \ldots, z_n, f_1, \ldots, f_k$ are l.i./$\mathbb{Q}$ mod $\mathbb{C}$.

The Ax-Lindemann-Weierstrass theorem for $(\mathbb{C}^\times)^n$ is the following particular case of Ax-Schanuel theorem.

Theorem 6.16 (Ax-Lindemann-Weierstrass, form 1)**.** *Let $W \subset \mathbb{C}^n$ an algebraic variety. Assume that the coordinates functions $z_1, \dots, z_n \in \mathbb{C}(W)$ are linearly independent over $\mathbb{Q}$ modulo $\mathbb{C}$. Then* $\exp(z_1), \dots, \exp(z_n)$ *are algebraically independent over $\mathbb{C}$.*

Proof. By Ax-Schanuel theorem 6.14,

$$\mathrm{tr.d.}_{\mathbb{C}}(z_1, \dots, z_n) + \mathrm{tr.d.}_{\mathbb{C}}(\exp(z_1), \dots, \exp(z_n)) \geq$$
$$\mathrm{tr.d.}_{\mathbb{C}}(z_1, \dots, z_n, \exp(z_1), \dots, \exp(z_n)) \geq n + \dim(W),$$

hence since $\mathrm{tr.d.}_{\mathbb{C}}(z_1, \dots, z_n) = \dim(W)$, $\mathrm{tr.d.}_{\mathbb{C}}(\exp(z_1), \dots, \exp(z_n)) = n$, *i.e.* there are algebraically independent over $\mathbb{C}$. $\square$

This theorem can be restated in more geometric terms. In the following statement, we say that a subvariety $Y \subset \mathbb{C}^n$ is geodesic if it defined by a set of non-trivial equations of the form $q_1 z_1 + \dots q_n z_n = c$, with $q_1, \dots, q_n \in \mathbb{Q}$ and $c \in \mathbb{C}$. This is equivalent to say that $\pi(Y)$ is a translate of a proper subgroup of $(\mathbb{C}^\times)^n$.

Theorem 6.17 (Ax-Lindemann-Weierstrass, form 2)**.** *Let $V \subset (\mathbb{C}^\times)^n$ be a strict closed algebraic subvariety of $(\mathbb{C}^\times)^n$ and let Y be a maximal irreducible complex algebraic variety contained in $\pi^{-1}(V)$. Then Y is a geodesic variety.*

Remark 6.18.
(1) Observe that this theorem is completely analogous to Theorem 6.10 we used in the proof of the Manin-Mumford conjecture.
(2) Ax-Schanuel theorem 6.14 can also be restated in a geometric form, see the article by Pila in [JW15].

Proof of the equivalence between the two forms. Assuming form 1, let V and $Y \subset \pi^{-1}(V)$ irreducible algebraic and maximal with this property. Choose a maximal subset $I \subset \{1, \dots, n\}$ such that $\exp(z_i)_{i \in I}$ are algebraically independent over $\mathbb{C}$. By Theorem 6.16, each z_j, for $j \notin I$ satisfies an equation of the form $z_j = \sum_{i \in I} q_{ij} z_i + c_j$. Let T be the geodesic variety defined by the above equations, for $j \notin I$. By construction $W \subset T$. Since the $\exp(z_i)_{i \in I}$ are algebraically independent over $\mathbb{C}$, $\pi(T) \subset V$. By maximality of W, we have $W = T$.

Assume now form 2, and suppose that $\exp(z_1), \dots, \exp(z_n)$ (viewed as functions on W) are not algebraically independent over $\mathbb{C}$. So $\pi(W) \subset V \subset \mathbb{C}^\times)^n$, for V a proper closed algebraic subvariety of $\mathbb{C}^\times)^n$. By Theorem 6.17, there is a geodesic variety W' with $W \subset W' \subset \pi^{-1}(V)$. So the $z_1, \dots, z_n$ are linearly dependent over $\mathbb{Q}$ modulo $\mathbb{C}$. $\square$

Exercise 6.19. Formulate and prove a functional version of the Ax-Lindemann-Weierstrass theorem for abelian varieties.

Exercise 6.20. Prove "Manin-Mumford for $(\mathbb{C}^\times)^n$". That is, prove that if $V \subset (\mathbb{C}^\times)^n$ is an irreducible subvariety defined over $\mathbb{Q}$ that does not contains any translate of a non-trivial subgroup of $(\mathbb{C}^\times)^n$, then $V \cap ((\mathbb{C}^\times)^n)^{\mathrm{tor}}$ is finite. Hint: follow the Pila-Zanier strategy: π restricted to a fundamental domain is definable in $\mathbb{R}_{\exp}$, and use Theorem 6.17 instead of Theorem 6.10. For the analog of Mazur's theorem, what can you say about the degree of a torsion point of $\mathbb{C}^\times$?

We now prove the last missing ingredient of our proof of the Manin-Mumford conjecture, the Ax-Lindemann-Weierstrass theorem for abelian varieties, recalled below. Note that the proof can be adapted to a proof of Theorem 6.17.

Theorem 6.21 (Ax-Lindemann-Weierstrass for abelian varieties)**.** *Let A be an abelian variety of dimension g over $\mathbb{C}$ and $\pi : \mathbb{C}^g \to A(\mathbb{C})$ the exponential map. Let V be a complex algebraic subvariety of A and Y a maximal irreducible complex algebraic subvariety contained in $\pi^{-1}(V)$. Then $\pi(Y)$ is a translate of an abelian subvariety of A.*

We follow the proof given by Orr in [JW15], following ideas by Pila, Ullmo and Yafaev. The proof will be organized as follows. We keep the notations introduced in the proof the Manin-Mumford conjecture, namely, $\Lambda = \ker(\pi)$, we identify $\mathbb{C}^g$ to $\mathbb{R}^{2g}$ using a basis of Λ, $F = [0,1[^{2g}$ is a fundamental domain of the action of Λ on $\mathbb{C}^g$. We want to apply the Pila-Wilkie theorem to the set

$$\Sigma = \left\{x \in \mathbb{C}^g \mid Y + x \cap F \neq \emptyset \text{ and } Y + x \subset \pi^{-1}(V)\right\}.$$

We have $\Sigma = \left\{x \in \mathbb{C}^g \mid Y + x \cap F \neq \emptyset \text{ and } Y + x \cap F \subset \pi_{|F}^{-1}(V)\right\}$. Indeed, $Y + x$ and $\pi^{-1}(V)$ are both analytic sets and $Y + x$ is irreducible. Hence since $Y + x \cap F \neq \emptyset$ and included in $\pi^{-1}(V)$, $Y + x \subset \pi^{-1}(V)$. Hence Σ is definable in $\mathbb{R}_{\mathrm{an}}$.

The strategy goes as follows.

Step 1: show that the number of points of $\Sigma \cap \Lambda$ of height at most H grows at least linearly in H.

Step 2: use the Pila-Wilkie theorem to find a infinite semi-algebraic set in Σ.

Step 3: deduce that the stabilizer of Y has positive dimension, and that its image via π is the stabilizer of the Zariski closure of $\pi(Y)$.

Step 4: conclude by applying again the argument to the quotient by that stabilizer.

We can assume that $\dim(Y) > 0$, otherwise the theorem is clear, and that V is the Zariski closure of $\pi(Y)$ in A (up to replacing V by the latter).

Lemma 6.22. *There exists some H_0 such that for every $H \geq H_0$, $\#\Sigma(\mathbb{Q}, H) \geq H/2$.*

Proof. Observe that since $Y \subset \pi^{-1}(V)$, $\Sigma \cap \Lambda = \{x \in \mathbb{C}^g \mid Y + x \cap F \neq \emptyset\}$. Since $\Lambda = \mathbb{Q}^{2g}$, we need to count points in the latter set of height at most H. Since Y is an irreducible affine algebraic variety, it is path connected and unbounded with respect to the Euclidean norm on $\mathbb{C}^g$. Hence we can find a continuous map $\gamma : [0, +\infty) \to Y$ with unbounded image. Each time the image of γ crosses the boundary between fundamental domains $F - x$ and $F - x'$ (with $x, x' \in \Lambda$), the heights of x and x' differs by at most 1. So the heights of points in $\Lambda_\gamma = \{x \in \Lambda \mid \mathrm{Im}(\gamma) \cap (F - x) \neq \emptyset\}$ form a set of consecutive integers. Since $\mathrm{Im}(\gamma)$ is unbounded, there is some h_0 such that for every integer $h \geq h_0$, Λ_γ contains a point of height h. Since $\Lambda_\gamma \subset \Sigma \cap \Lambda$, the lemma is proved with $H_0 = 2h_0$. □

By the Pila-Wilkie theorem 5.2, there is some connected irreducible positive dimensional semi-algebraic set $W \subset \Sigma$ and one can further assume that W contains a point $w_0 \in \Lambda$.

Let $\Theta \subset \mathbb{C}^g$ be the stabilizer of Y (which is a $\mathbb{C}$-vector subspace of $\mathbb{C}^g$) and $B \subset A$ the identity component of the stabilizer of V (an abelian subvariety of A). We will show that both Θ and B have positive dimension by showing first that $W - w_0 \subset \Theta$, then that $\pi(\Theta) = B$.

We need to show that $Y + W - w_0 = Y$. Since $W \subset \Sigma$, $Y + W \subset \pi^{-1}(V)$. Since $\pi^{-1}(V)$ is Λ-invariant, we have $Y + W - w_0 \subset \pi^{-1}(V)$. Since $Y + W - w_0$ is a connected irreducible real semi-algebraic set, by Lemma 6.11, there is some irreducible complex algebraic variety Y' such that $Y \subset Y + W - w_0 \subset Y' \subset \pi^{-1}(V)$. By maximality of Y, $Y = Y + W - w_0 = Y'$.

We now show that $\pi(\Theta) = B$. Fix $x \in \Theta$. We have $Y + x = Y \subset \pi^{-1}(V)$, hence $Y \subset \pi^{-1}(V) - x$. We deduce that $\pi(Y) \subset V \cap (V - \pi(x))$. Since V is the Zariski closure of $\pi(Y)$, we have that $V = V - \pi(x)$, hence $\pi(\Theta)$ stabilize V. Since Θ is connected, $\pi(\Theta)$ is connected in the Euclidean topology hence in the Zariski topology, so $\pi(\Theta) \subset B$.

It remains to show that $B \subset \pi(\Theta)$. Let Θ' be the identity component of $\pi^{-1}(B)$ in the Euclidean topology. Note that $\pi(\Theta')$ is an analytic subgroup of B, with the same dimension as B, so is equal to B. Hence it is enough to show that $\Theta' \subset \Theta$. We have $Y + \Theta' \subset \pi^{-1}(V)$. But $Y + \Theta'$ is an irreducible complex algebraic variety containing Y, hence $Y + \Theta' = Y$, *i.e.* $\Theta' \subset \Theta$.

To conclude the proof of the theorem, we consider the quotient abelian variety $A' = A/B$, with the quotient map $q : A \to A/B$. We also have a quotient map $q' : \mathbb{C}^g \to \mathbb{C}^g/\Theta$. Since $B = \pi(\Theta)$, we have a commutative diagram

$$\begin{array}{ccc} \mathbb{C}^g & \xrightarrow{q'} & \mathbb{C}^g/\Theta \\ {\scriptstyle\pi}\big\downarrow & & \big\downarrow{\scriptstyle\pi'} \\ A & \xrightarrow{q} & A', \end{array}$$

where π' is the exponential map for A'. Set $V' = q(V)$ and $Y' = q'(Y)$. The maximality of Y implies that Y' is a maximal irreducible algebraic subvariety of $\pi'^{-1}(V')$.

If $\dim(Y') > 0$, we can apply what we have done so far to (A', V', Y') to find that the stabilizer of V' in A' has positive dimension. But its preimage by q stabilize V, contradicting the fact that B is the stabilizer of V. Hence Y' is a point, which means that $\pi(Y)$ is a translate of the abelian subvariety B.

6.4 The André-Oort conjecture for a product of modular curves

So far we have seen that the Pila-Zannier strategy can be applied to the Manin-Mumford problem in two different contexts : the uniformizing map $\pi : \mathbb{C}^g \to A$ of an abelian variety and the exponential map $\mathbb{C}^n \to (\mathbb{C}^\times)^n$. There is a third situation where one can apply the same strategy, which is the André-Oort conjecture for Shimura varieties. We will only give details in the particular case of product of modular curves. In order to state the result, we need to introduce the modular curve, which is the moduli space of elliptic curves. We only state the results we need, refering to Silverman [Sil09] and [Sil94, Chapters 1 and 2] for details and proofs.

In this section we only consider elliplic curves E defined over (a subfield of) $\mathbb{C}$, and we identify E with its sets of complex points. Recall that an elliplic cuve E defined over $\mathbb{C}$ is isomorphic (as a complex Lie group) to $\mathbb{C}/\Lambda$, where $\Lambda \subset \mathbb{C}$ is a lattice.

Since a lattice is of the form $\Lambda = \mathbb{Z}\omega_1 + \mathbb{Z}\omega_2$, with $\omega_1, \omega_2 \in \mathbb{C}$ $\mathbb{R}$-linearly independent, up to multiplying Λ by some $\alpha \in \mathbb{C}*$, we see that any elliptic curve over $\mathbb{C}$ is of the form $E_\tau = \mathbb{C}/\Lambda_\tau$, where $\Lambda_\tau = \mathbb{Z} + \mathbb{Z}\tau$ with $\tau \in \mathbb{H} = \{z \in \mathbb{C} \mid \mathrm{Im}(z) > 0\}$.

The group $\mathrm{SL}_2(\mathbb{R})$ acts on $\mathbb{H}$ via Moebius transformations:

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \cdot z := \frac{az+b}{cz+d}.$$

You can check that E_τ and $E_{g\tau}$ are isomorphic if and only if $g \in \mathrm{SL}_2(\mathbb{Z})$. Hence the quotient $\mathrm{SL}_2(\mathbb{Z})\backslash\mathbb{H}$ parametrizes all complex elliptic curves up to isomorphism. A (the closure of a) fundamental domain of the action of $\mathrm{SL}_2(\mathbb{Z})$ is given by

$$\bar{\mathcal{F}} = \left\{ \tau \in \mathbb{H} \mid \Re(\tau) \leq \frac{1}{2}, |\tau| \geq 1 \right\}.$$

We will see that $\mathrm{SL}_2(\mathbb{Z})\backslash\mathbb{H}$ admits an algebraic structure, by introducing the modular function j. Recall that a complex elliptic curve E admits a Weierstrass equation of the form $y^2 = x^3 + ax + b$, where $\Delta(E) := -16(4a^3 + 27b^2) \neq 0$. Define the j-invariant of the elliptic curve E by

$$j(E) = \frac{-1728(4a)^3}{\Delta(E)}.$$

View j as a map $j : \mathbb{H} \to \mathbb{C} = \mathbb{A}^1(\mathbb{C})$ via $j(\tau) = j(E_\tau)$.

Theorem 6.23. *The map $j : \mathbb{H} \to \mathbb{C}$ is holomorphic, surjective, and satisfies $j(g\tau) = j(\tau)$ if and only if $g \in \mathrm{SL}_2(\mathbb{Z})$. That is, j induces a complex analytic isomorphism $\mathrm{SL}_2(\mathbb{Z})\backslash\mathbb{H} \simeq \mathbb{C}$.*

Sketch of proof. The holomorphicity of j comes from the theory of elliptic functions. For the invariance, observe that if E and E' have respective Weierstrass equations $y^2 = x^3 + ax + b$ and $(y')^2 = (x')^3 + a'x' + b'$, then an isomorphism between E and E' must be of the form $x = u^2x'$ and $y = u^3y'$, for some $u \in \mathbb{C}^\times$. So if E and E' are isomorphic, $a = u^4a'$, $b = u^6b'$, $\Delta(E) = u^{12}\Delta(E')$ and $j(E) = j(E')$.

Reciprocally, if $j(E) = j(E')$, then one has $a^3(b')^2 = (a')^3b^2$. One find the desired parameter u defining the isomorphism by a case by case analysis depending on whether a or b is zero (equivalently whether $j = 0, 1728$ or neither of those). For example, if $ab \neq 0$, one can take $u = \left(\frac{a}{a'}\right)^{1/4} = \left(\frac{b}{b'}\right)^{1/6}$.

For the surjectivity, if $j_0 \in \mathbb{C}\setminus \{0, 1728\}$, then the curve E of equation

$$y^2 + xy = x^3 - \frac{36}{j_0 - 1728}x - \frac{1}{j_0 - 1728}$$

has j-invariant equal to j_0. The remaining cases correspond for example to curves of equations respectively $y^2 + y = x^3$ and $y^2 = x^3 + x$. $\square$

We call $\mathbb{C}$, together with the map $j : \mathbb{H} \to \mathbb{C}$, the *modular curve*, since it is the moduli space of elliptic curves.

If one want to pursue the analogy with the Manin-Mumford conjecture, one needs a notion of special point (formerly the torsion points) and special subvariety (formerly the torsion cosets of abelian subvarieties).

For the special points, we need to identify a class of elliptic curves with special properties. They will be provided by the theory of complex multiplication. Recall that since an elliptic curve is a group, its ring of endomorphisms $\mathrm{End}(E)$ always contain a copy of $\mathbb{Z}$, where $n \in \mathbb{Z}$ is viewed as the multiplication by n in E. Many elliptic curves have endomorphism rings isomorphic to $\mathbb{Z}$.

We say that an elliptic curve E has *complex multiplication* (or that E is a *CM-elliptic curve*) if $\mathrm{End}(E) \neq \mathbb{Z}$, that is, if E has an endomorphism that is not the multiplication by an integer.

Proposition 6.24. *It E_τ has complex multiplication, then $\mathbb{Q}(\tau)$ is a quadratic imaginary field and $\mathrm{End}(E_\tau)$ is an order in $\mathbb{Q}(\tau)$, that is, a finite rank $\mathbb{Z}$-module such that $\mathrm{End}(E_\tau) \otimes \mathbb{Q} = \mathbb{Q}(\tau)$.*

Proof. Using covering space theory, one can show that

$$\mathrm{End}(E_\tau) \simeq \{\alpha \in \mathbb{C} \mid \alpha\Lambda_\tau \subset \Lambda_\tau\}.$$

Hence for any $\alpha \in \mathrm{End}(E_\tau)$, we have $\alpha = a + b\tau$ and $\alpha\tau = c + d\tau$. By solving for α, we find $\alpha^2 - (a+d)\alpha\tau - c = 0$, hence $\mathrm{End}(E_\tau)$ is an integral extension of $\mathbb{Z}$. If $\alpha \in \mathrm{End}(E_\tau)\backslash\mathbb{Z}$, then $b \neq 0$ and we can solve for τ and find that $b\tau^2 - (a-d)\tau - c = 0$, hence $\mathbb{Q}(\tau)$ is a quadratic imaginary extension of $\mathbb{Q}$ (since $\tau \notin \mathbb{R}$). Finally $\mathrm{End}(E_\tau) \otimes \mathbb{Q} = \mathbb{Q}(\tau)$ from degree considerations. $\square$

The class group $\mathrm{Cl}(\mathcal{R})$ of an order $\mathcal{R}$ in a quadratic imaginary field K is the group of fractional ideals of $\mathcal{R}$ modulo principal ideals. It is finite, and we call its cardinal the class number of $\mathcal{R}$. If $\mathcal{R}$ is the ring of integers of K, this is the class number of K.

If $\mathcal{R}$ is the endomorphism ring of an elliptic curve, then there is a bijection between $\mathrm{Cl}(\mathcal{R})$ and the isomorphism classes of elliptic curves E such that $\mathrm{End}(E) \simeq \mathcal{R}$. Note that it implies that if E is a CM-elliptic curve, $j(E)$ is algebraic over $\mathbb{Q}$.

The main theorem of complex mutliplication (which uses class field theory) implies that for a CM elliptic curve E, $[Q(j(E) : \mathbb{Q}] = \#\mathrm{Cl}(\mathrm{End}(E))$.

Hence we can use the following lower bounds for $\#\mathrm{Cl}(\mathrm{End}(E))$ to find lower bounds for $[Q(j(E) : \mathbb{Q}]$, which will be replace Masser's theorem in the Pila-Zannier strategy.

Let E_τ be a CM-elliptic curve with $\tau \in \mathcal{F}$ and $D(\tau)$ be the discriminant of τ, *i.e.* $D(\tau) = b^2 - 4ac$ where $aX^2 + bX + c$ is the minimal polynomial of τ. Siegel has proven that for every $\eta > 0$, there is some c_η such that

$$\#\mathrm{Cl}(\mathrm{End}(E_\tau)) \geq c_\eta \left|D(\tau)\right|^{1/2-\eta}.$$

We call a point $(c_1, \ldots, c_n) \in \mathbb{A}^n(\mathbb{C})$ *special* if each c_i is the j-invariant of a CM-elliptic curve. To describe the special subvarieties of a product of modular curves, we need the notion of modular polynomial.

Let $\mathrm{GL}_2^+(\mathbb{Q})$ be the subset of $\mathrm{GL}_2^+(\mathbb{Q})$ constituted of matrix with positive determinant. For $g \in \mathrm{GL}_2^+(\mathbb{Q})$, we can rescale it such that its enties are in $\mathbb{Z}$ and coprime.

We set $N(g)$ to be the determinant of this rescaled matrix. The N-th modular polynomial is an irreducible polynomial $\Phi_N \in \mathbb{Z}[X,Y]$ such that $\Phi_N(j(\tau), j(g\tau)) = 0$ for any $\tau \in H$ and $g \in \mathrm{GL}_2^+(\mathbb{Q})$ such that $N(g) = N$. It is symetric for $N > 1$ and $\phi_1(x,y) = x - y$. The curve defined by Φ_N also related to a moduli space. Its desingularisation is complex analytically isomorphic to the quotient $\Gamma_0(N)\backslash\mathbb{H}$, where

$$\Gamma_0(N) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z}) \mid c \equiv 0 \mod N \right\}.$$

Observe that the function $j_N : \tau \in \mathbb{H} \mapsto j(N\tau)$ is precisely invariant under the group $\Gamma_0(N)$. The polynomial Φ_N is defined as the unique monic polynomial such that $\Phi_N(j(\tau), j(\tau')) = 0$ if and only if there is an isogeny (*i.e.* a non-trivial morphism) between E_τ and $E_{\tau'}$ with kernel cyclic of order N.

A *special subvariety* of $\mathbb{A}^n_{\mathbb{C}}$ is a subvariety defined by a system of equations of the form $\Phi_{N_{i,j}}(x_i, x_j) = 0$ and $x_i = c_i$ where c_i is a special point. A *weakly special subvariety* is defined by the same system of equations, but without requiring the c_i to be special points.

With these definitions, we can now state the André-Oort conjecture for product of modular curves, which was proven by Pila in [Pil11].

Theorem 6.25. *Let $Y \subset \mathbb{A}^n_{\mathbb{C}}$ be an irreducible subvariety such that the set of special points in $Y(\mathbb{C})$ is Zariski-dense in Y. Then Y is a special subvariety.*

Remark 6.26. Pila proves in fact a slightly more general result, replacing $\mathbb{A}^n_{\mathbb{C}}$ by $\mathbb{A}^n_{\mathbb{C}} \times E_1 \cdots \times E_k \times {\mathbb{G}_m}^m$, where E_i is an elliptic curve and special points and sub varieties of $E_1 \cdots \times E_k \times {\mathbb{G}_m}^m$ are defined respectively as torsion points and torsion coset of a subgroup.

The strategy of proof is the same as in the Manin-Mumford case. By the same kind of consideration used at the end of the proof of the Manin-Mumford conjecture, we need to show that if $Y \subset \mathbb{A}^n_{\mathbb{C}}$ is an irreducible subvariety defined over $\mathbb{Q}$ that contains no weakly special subvariety, then Y contains only finitely many special points.

We want to put this situation in an o-minimal setting. Since $j(\tau) = j(\tau + 1)$, it admits a Fourier expansion in the variable $q = \exp(2\pi i\tau)$. One can show that this expansion has the form

$$j(\tau) = q^{-1} + 744 + \sum_{n=0}^{+\infty} a_n q^n.$$

It follows that the map j, restricted to the fundamental domain $\mathcal{F}$ is definable in $\mathbb{R}_{\text{an,exp}}$. Indeed, the map $\tau \mapsto q = \exp(2\pi i\tau)$ restricted to $\mathcal{F}$ is definable in $\mathbb{R}_{\text{an,exp}}$, since one need to define it the unbounded real exponential map and the maps sin and cos restricted to some bounded domains. Since its image is an open disc, the map $q \mapsto q^{-2} + 722 + \sum_{n=0}^{+\infty} a_n q^n$ (restricted to this disc) is definable in $\mathbb{R}_{\text{an}}$. Hence the map $\pi : (z_1, \ldots, z_n) \in \mathbb{H}^n \mapsto (j(z_1), \ldots, j(z_n)) \in \mathbb{C}^n$, restricted to the fundamental domain $\mathcal{F}^n$ is definable in the o-minimal structure $\mathbb{R}_{\text{an,exp}}$.

Contrary to the Manin-Mumford case, the preimage by π of a special point is not a rational point, but an algebraic point (actually a quadratic point). Hence if we want to use the Pila-Wilkie theorem, we need to extend it to algebraic numbers. We define the height $H(x)$ of an algebraic number $x \in \mathbb{Q}^{\text{alg}}$ as the maximum of the height of the coefficients of its minimal polynomial over $\mathbb{Q}$. Theorem 5.2 admits the following generalization (see [Pil09]):

Theorem 6.27. *Let $X \subset \mathbb{R}^n$ be a set definable in an o-minimal expansion of $(\mathbb{R}, <, +, -, \cdot)$. Then for every $k \geq 1$ and $\varepsilon > 0$, there is a constant $C = C(k, X, \varepsilon)$ such that for every $H \geq 1$,*

$$\# X^{\text{tran}}(\mathbb{Q}^{\text{alg},\deg\leq k}, H) \leq C H^{\varepsilon},$$

where $\mathbb{Q}^{\text{alg},\deg\leq k}$ is the set of algebraic points of degree at most k.

We use this theorem with $k = 2$ and $X = \pi^{-1}_{|\mathcal{F}^n}(Y)$. We want to compare the obtained bound against the lower bound provided by Siegel's class number formula (for $\tau \in \mathbb{H}$ corresponding to a CM-elliptic curve):

$$[Q(j(\tau) : \mathbb{Q}] = \#\mathrm{Cl}(\mathrm{End}(E_\tau)) \geq c_\eta \left|D(\tau)\right|^{1/2-\eta}.$$

To compare the height and discriminant, we use the elementary fact that there is an absolute constant K such that $H(\tau) \leq KD(\tau)$ for every $\tau \in \mathcal{F}$ corresponding to a CM-elliptic curve.

Comparing the two bounds, we see that there are only finitely many points in $X^{\text{tran}}(\mathbb{Q}^{\text{alg},\deg\leq 2}$. Indeed, if $x \in X^{\text{tran}}(\mathbb{Q}^{\text{alg},\deg\leq 2}$, then $\pi(x) \in Y$ is a special point, and all its Galois conjugates are also special points in Y. By Siegel's class number formula,

$$[Q(\pi(x)) : \mathbb{Q}] \geq K' H(x)^{1/2-\eta}.$$

Hence $H(x)$ must be bounded, otherwise it would contradict Pila-Wilkie's bound from theorem 6.27.

To conclude the proof, it remains to identify the algebraic part of X. This is done using the following version of the Ax-Lindemann-Weierstrass theorem, due to Pila. We say that $W \subset \mathbb{H}^n$ is geodesic if is defined by a system of equations of the form $z_j = g_{ij} z_i$ and $z_i = c_i$, with $g_{ij} \in \mathrm{GL}_2^+(\mathbb{Q})$ and $c_i \in \mathbb{H}$. Equivalently, $\pi(W)$ is a weakly special subvariety of $\mathbb{C}^n$.

Theorem 6.28 (modular Ax-Lindemann-Weierstrass)**.** *Let $\pi = (j, \ldots, j) : \mathbb{H}^n \to \mathbb{C}^n$. Let V be a complex algebraic subvariety of $\mathbb{A}^n_{\mathbb{C}}$ and Y a maximal irreducible complex algebraic subvariety contained in $\pi^{-1}(V)$. Then Y is geodesic, equivalently, $\pi(Y)$ is a weakly special subvariety.*

Contrary to statements for $\mathbb{G}_{\mathrm{m}}^n$ and abelian varieties discussed in Section 6.3, there is no differential algebraic proof of the functional version of this theorem. It was proven by Pila using o-minimality and a strategy similar to the proof of Ax-Lindemann-Weierstrass of Abelian varieties we exposed earlier.

The general André-Oort conjecture is concerned with Shimura varieties, which are algebraic varieties X, together with a complex analytic uniformizing map $U \to X(\mathbb{C})$. The prototypical example is modular curves and more generaly modular varieties $\mathcal{A}_g$, which are moduli space of principally polarized abelian varieties of dimension g. Those admits also a notion of special points and subvarieties, and the André-Oort conjecture is the statement that an irreducible subvariety of a Shimura variety that contains a Zariski-dense set of special points is a special subvariety.

Beside the case of product of modular curves proved by Pila, as discussed above, the conjecture was proved for $\mathcal{A}_g$ ($g \leq 6$) by Pila and Tsimerman in [PT14]. To prove the general case, one can try to apply the Pila-Zannier strategy, but some ingredients are missing. By Peterzil and Starchenko [PS13] and Klingler, Ullmo and Yafaev [KUY16] is known that the uniformizing map of a Shimura variety restricted to a fundamental domain is definable in $\mathbb{R}_{\mathrm{an,exp}}$. The version of the Ax-Lindemann-Weierstrass theorem in this situation is also now proven in general, by Ullmo and Yafaev [UY14b] and Klingler, Ullmo and Yafaev [KUY16]. The main missing piece is the lower bound for Galois orbits of special points. Assuming the generalized riemann hypothesis, Ullmo and Yafaev [UY14a] have proved the required bounds.

Let us conclude by mentioning a broader conjecture, the Zilber-Pink conjecture, which unifies the Manin-Mumford problem for $\mathbb{G}_{\mathrm{m}}^n$ and abelian varieties, the André-Oort conjecture for mixed Shimura varieties, as well as the Mordell-Lang conjecture. If $V \subset X$ is an irreducible subvariety of X, where X is either an (semi-)abelian variety or a mixed Shimura variety, then we say that a variety A is an atypical subvariety of X if there is a special subvariety $T \subset X$ such that A is an irreducible component of $T \cap X$ such that

$$\dim(A) > \dim(V) + \dim(T) - \dim(X).$$

Then the Zilber-Pink conjecture states that V contains only finitely many maximal atypical subvarieties.

Oberve that the Zilber-Pink conjecture implies André-Oort conjecture. Indeed, assume that V is irreducible and contains a Zariski-dense set of special points, since a special point is a special subvariety, all special points of V are atypical subvarieties of V, hence they must be contained in a single atypical suvariety equal to V, *i.e.* V is special.

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