

**RANDOM WALKS ON TRANSITIVE GRAPHS (D-MATH)  
EXERCISE SHEET 10**

In the following exercises we consider  $G = (V, E)$  an infinite, locally finite, connected, and transitive graph of degree  $d > 0$  with fixed origin  $o \in V$ . Let  $(X_n)$  be the lazy random walk on  $G$  started at  $o$ .

(★) **Exercise 1.** Let  $H(X_n)$  be the entropy of the lazy random walk on  $G$ .

(i) Show that for all  $n, m \geq 0$ ,  $H(X_{n+m}) \leq H(X_n) + H(X_m)$ .

(ii) Conclude that

$$h := \lim_{n \rightarrow \infty} \frac{H(X_n)}{n}.$$

exists. We call  $h$  the *Avez entropy* of  $G$ .

(★) **Exercise 2.**

(i) Prove that  $h \leq \log(d)$ .

(ii) Prove that  $h = 0$  if  $G$  has exponential growth.

(iii) Prove that  $h > 0$  if  $G = \mathbb{T}_3$ , the 3-regular tree.

(★) **Exercise 3.** Let  $\rho(G)$  be the spectral radius of the lazy random walk on  $G$ . Show that

$$h \geq -2 \log(\rho(G)).$$