ETH Zürich, HS 2022
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RANDOM WALKS ON TRANSITIVE GRAPHS (D-MATH)
EXERCISE SHEET 11

($\star$) **Exercise 1.** (Total variation distance) Let $\mathcal{X}$ be a finite set and μ, ν two probability measures on $\mathcal{X}$. We define the *total variation distance* between μ and ν by

$$d_{\mathrm{TV}}(\mu,\nu) := \max_{A\subset\mathcal{X}} |\mu(A)-\nu(A)|.$$

(i) Prove that

$$d_{\mathrm{TV}}(\mu,\nu) = \frac{1}{2}\sum_{x\in\mathcal{X}} |\mu(x)-\nu(x)|.$$

(ii) Show that

$$d_{\mathrm{TV}}(\mu,\nu) = 1-\sum_{x\in\mathcal{X}} \mu(x)\wedge\nu(x).$$

(iii) Let $p := 1-d_{\mathrm{TV}}(\mu,\nu)$. Assume that $p\in(0,1)$. Consider X, Y, Z, B independent random variables with

$$X \sim \frac{1}{p}(\mu\wedge\nu),\quad Y\sim\frac{1}{1-p}(\mu-\mu\wedge\nu),\quad Z\sim\frac{1}{1-p}(\nu-\mu\wedge\nu),\quad B\sim \mathrm{Ber}(p).$$

Let us define

$$(U_0,V_0) := \begin{cases} (X,X) & \text{if } B=1,\\ (Y,Z) & \text{if } B=0.\end{cases}$$

Check that X, Y, Z are well defined random variables, and prove that $U_0\sim\mu$, $V_0\sim\nu$, and $d_{\mathrm{TV}}(\mu,\nu) = \mathbb{P}[U_0\neq V_0]$.

(iv) Prove that

$$d_{\mathrm{TV}}(\mu,\nu) = \min_{\substack{U\sim\mu\\ V\sim\nu}} \mathbb{P}[U\neq V],$$

where the minimum is taken over all random variables U, V with $U\sim\nu$ and $V\sim\nu$.

($\star$) **Exercise 2.** (Sedrakyan's inequality) Let X, Y be real random variables with $Y>0$. Prove that

$$\mathbb{E}\left[\frac{X^2}{Y}\right] \geq \frac{\mathbb{E}[|X|]^2}{\mathbb{E}[Y]}.$$

Hint: Use Cauchy-Schwarz.

($\star$) **Exercise 3.** Let $\mathcal{X}$ be a finite set, and X, Y be two random variables with values on $\mathcal{X}$. Let $\widetilde{X}$, $\widetilde{Y}$ be two independent copies of X and Y, that is

$$\widetilde{X}\sim X,\ \widetilde{Y}\sim Y, \text{ and } \widetilde{X}\perp\!\!\!\perp\widetilde{Y}.$$

Let us define

$$r(x,y) := \frac{\mathbb{P}[X=x, Y=y]}{\mathbb{P}[\widetilde{X}=x,\widetilde{Y}=y]} = \frac{\mathbb{P}[X=x,Y=y]}{\mathbb{P}[X=x]\mathbb{P}[Y=y]},$$

with the convention $0/0 = 1$. Consider the random variable $R := r(\widetilde{X}, \widetilde{Y})$.

(i) Check that $\mathbb{E}[R] = 1$ and $\mathbb{E}[R \log R] = H(X) + H(Y) - H(X,Y)$.

(ii) Prove that $\forall u \geq 0$,

$$u \log u - u + 1 \geq \frac{3}{2} \frac{(u-1)^2}{2+u}.$$

(iii) Prove that

$$\mathbb{E}[R \log R] \geq \frac{3}{2} \frac{\mathbb{E}[|R-1|]^2}{\mathbb{E}[2+R]}.$$

(iv) Conclude that

$$\sum_{x,y \in \mathcal{X}} |\mathbb{P}[X = x, Y = y] - \mathbb{P}[X = x]\mathbb{P}[Y = y]| \leq \sqrt{2}\sqrt{H(X) + H(Y) - H(X,Y)}.$$

($\star$) Exercise 4. Let (X_n) be the lazy random walk on a connected and infinite transitive graph $G = (V, E)$. Let us assume that its Avez entropy $h = 0$.

(i) Prove that $\forall k \geq 0$,

$$H(X_{k+n}) + H(X_k) - H(X_n) \xrightarrow[n \to \infty]{} 0.$$

(ii) Deduce that $\forall x, y \in V$,

$$d_{\mathrm{TV}}(\mu_x^n, \mu_y^{n-d}) \xrightarrow[n \to \infty]{} 0,$$

where $d = d(x,y)$ and μ_x^n is the law of X_n under $\mathbb{P}_x$ (i.e. $\mu_x^n(z) = p_n(x.y)$).