ETH Zürich, HS 2022
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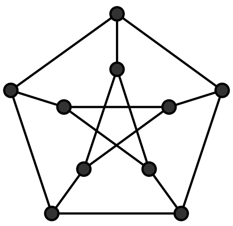
RANDOM WALKS ON TRANSITIVE GRAPHS (D-MATH)
EXERCISE SHEET 2

In this exercise sheet we consider $G = (V, E)$ an infinite, locally finite, connected, and transitive graph with fixed origin $o \in V$.

($\star$) Exercise 1. Show that $\forall x, y \in V,\ B_n(x) \cong B_n(y)$.

Exercise 2. We consider Petersen's graph, defined by

$G' \quad =$ 

(i) Check that G' is transitive.
(ii) Prove that G' is not a Cayley graph.

Hint: use that the only groups of order 10 *are* $\mathbb{Z}/10\mathbb{Z}$ *and the dihedral group* D_{10}.

($\star$) Exercise 3. Let (X_n) be the simple random walk on G, starting at o. Let (Z_i) be a sequence of i.i.d. random variables with distribution $\mathrm{Ber}(1/2)$. We define

$$T_0 = 0, \quad T_n = \sum_{i=1}^{n} Z_i \text{ for } n \geq 1, \quad \text{and } \widetilde{X}_n = X_{T_n} \text{ for } n \geq 0.$$

Prove that $(\widetilde{X}_n)$ is a lazy random walk starting from x.

($\star$) Exercise 4. For every $x \in V$ we fix $\phi_x \in \mathrm{Aut}(G)$ such that $\phi_x(o) = x$. Let $(Z_i)_{i\geq 1}$ be an i.i.d. sequence of random variables with uniform distribution in $\mathcal{N}_o = \{y \in V : y \sim o\}$. We define

$$X_0 = x, \quad X_{n+1} = \phi_{X_n}(Z_{n+1}) \text{ for } n \geq 1.$$

Show that (X_n) is a simple random walk on G started at x.

Exercise 5. (Fekete's Subadditivity Lemma) Let $(u_n)_{n\geq 0}$ be a subadditive sequence in $[-\infty, +\infty)$, that is

$$\forall m, n \geq 0, u_{m+n} \leq u_m + u_n.$$

Prove that the limit of $(\frac{u_n}{n})$ exists in $[-\infty, +\infty)$ and it is equal to $\inf_{n\geq 0} \frac{u_n}{n}$.