ETH Zürich, HS 2022
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RANDOM WALKS ON TRANSITIVE GRAPHS (D-MATH)
EXERCISE SHEET 3

In this exercise sheet we consider $G = (V, E)$ an infinite, locally finite, connected, and transitive graph of degree $d > 0$ with fixed origin $o \in V$.

($\star$) **Exercise 1.** Let (X_n) be a simple random walk on G, starting at o. Show that (X_n) is aperiodic if and only if G is not bipartite.

In the following exercises we denote by $p_n(o) := \mathbb{P}_o[X_n = o]$, $n \geq 0$ the return probability of (X_n) to o after n steps, where (X_n) is either a simple or a lazy random walk on G starting at o.

($\star$) **Exercise 2.**

(i) Prove that $p_{2n}(o)$ is non increasing in n.
(ii) Prove that $p_{2n+1}(o) \leq p_{2n}(o)$ for all $n \geq 0$.
(iii) Is $p_{2n+1}(o)$ decreasing in n?

($\star$) **Exercise 3.** Prove that $p_{2n}(o) \geq \dfrac{1}{|B_{2^n}|}$, for every $n \geq 0$.

($\star$) **Exercise 4.** Let (X_n) and $(\widetilde{X}_n)$ be respectively a simple and a lazy random walk on G, starting at o.

(i) Is it true that $\mathbb{P}_o[\widetilde{X}_{2n} = o] \geq \mathbb{P}_o[X_{2n} = o]$ for every $n \geq 0$?
(ii) Show that $\mathbb{P}_o[\widetilde{X}_{2n} = o] \geq \frac{1}{2}\mathbb{P}_o[X_{2n} = o]$ for every $n \geq 0$.

Exercise 5.

(i) Prove that if (X_n) is a lazy random walk, then

$$p_{2n+1}(o) \sim p_{2n}(o) \text{ as } n \to \infty.$$

(ii) Suppose that (X_n) is an aperiodic simple random walk. Show that

$$p_{2n+1}(o) \sim p_{2n}(o) \text{ as } n \to \infty.$$

Exercise 6. Let $S \subset V$ finite. We recall the following definitions

$$\begin{aligned} \partial S &= \{\{x,y\} \in E : x \in S,\ y \in V \setminus S\}, \\ \partial^{\text{in}} S &= \{x \in S : \exists y \in V \setminus S,\ y \sim x\}, \\ \partial^{\text{out}} S &= \{x \in V \setminus S : \exists y \in S,\ y \sim x\}. \end{aligned}$$

(i) Prove that $|\partial^{\text{in}} S| \leq |\partial S| \leq (d-1)|\partial^{\text{in}} S|$.
(ii) Is it always true that $|\partial^{\text{in}} S| \leq |\partial^{\text{out}} S|$?

($\star$) **Exercise 7.** Let $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ be infinite, locally finite and transitive graphs. We define the product graph $G_1 \times G_2 := (V, E)$ where $V = V_1 \times V_2$ and

$$E = \big\{\{\{x_1, x_2\}, \{y_1, y_2\}\} : (x_1 = y_1, \{x_2, y_2\} \in E_2) \text{ or } (x_2 = y_2, \{x_1, y_1\} \in E_1)\big\}.$$

Show that $\Phi(G_1 \times G_2) = \Phi(G_1) + \Phi(G_2)$, where Φ is the isomperimetric constant of the given graph.