RANDOM WALKS ON TRANSITIVE GRAPHS (D-MATH)
EXERCISE SHEET 4

In this exercise sheet we consider $G = (V, E)$ an infinite, locally finite, connected, and transitive graph of degree $d > 0$ with fixed origin $o \in V$. We denote $p_n(o) := \mathbb{P}_o[X_n = o]$, $n \geq 0$ the return probability of (X_n) to o after n steps, where (X_n) is a lazy random walk on G starting at o.

Definition (Grandparent Tree). Let T be a regular tree of degree 3. An *end* of T is an equivalence class of *rays* (that is, infinite, simple paths), where rays may start from any vertex and two rays are equivalent if they share all but finitely many vertices. Thus, given an end ξ, for every vertex x in T, there is a unique ray $x_\xi := (x_0, x_1, x_2, ...)$ in the class ξ that starts from $x = x_0$ (exercise). Call x_2 in this ray the ξ-*grandparent* of x. We call *grandparent tree* to the graph obtained from T by adding, for every x, an edge $\{x, x_2\}$ between x and its ξ-grandparent.

Remark: This graph is transitive and has degree 8 (exercise).

($\star$) **Exercise 1.** Let G be the grandparent tree given by an end ξ. Consider the evolving sets (S_n) associated to (X_n) on this graph G, with $S_0 = \{o\}$.

(i) Show that there exists n such that $\mathbb{P}[S_n \neq \emptyset, o \notin S_n] > 0$.

(ii) Let x_ξ the unique ray of ξ that starts at o. Show that for all $x \in x_\xi$ there exists n large enough such that $\mathbb{P}[S_n = \{x\}] > 0$.

($\star$) **Exercise 2.** Consider the evolving sets (S_n) associated to the lazy random walk on $\mathbb{Z}^2$ (starting from $S_0 = \{(0, 0)\}$). Compute

$$|\{S \subset V : \mathbb{P}[S_n = S] > 0\}|.$$

($\star$) **Exercise 3.** Suppose that G has exponential growth. Show that there exists a constant $c > 0$ such that for all $u \geq 1$,

$$\varphi(u) \geq \frac{c}{\log(2u)},$$

where φ denotes the expansion profile of G. Use this to prove that there exist constants $c_1, c_2 > 0$ such that for all $n \geq 0$,

$$p_n(o) \leq c_1 \exp(-c_2 n^{1/3}).$$

Exercise 4. (Mass-Transport Principle) Let us assume that G is amenable, and consider $f : V \times V \to [0, \infty]$ an invariant map, that is, for every $x, y \in V$ and every $\phi \in \mathrm{Aut}(G)$, $f(\phi(x), \phi(y)) = f(x, y)$.

(i) We assume that $||f||_\infty < \infty$ and that there exists $n < \infty$ such that for every $x, y \in V$ with $d(x, y) \geq n$, we have $f(x, y) = 0$. Let $K \subset V$ finite. Prove that

$$\left| \sum_{\substack{x \in K \\ y \in V}} f(x, y) - \sum_{\substack{x \in V \\ y \in K}} f(x, y) \right| \leq 2||f||_\infty |B_n|^2 |\partial K|.$$

Deduce that

$$\sum_{x \in V} f(o, x) = \sum_{x \in V} f(x, o). \tag{1}$$

(ii) Prove (1) for a general invariant $f : V \times V \to [0, \infty]$.

Exercise 5. In this exercise we want to prove the following theorem for G amenable

Theorem. *Define for every $m \geq 1$, $R(m) := \min\{r : |B_r| \geq m\}$. For every $u \geq 1$*

$$\varphi(u) \geq \frac{1}{2R(2u)}.$$

In order to do so, let us fix an integer $r \geq 0$. For every $x \in V$, we define

$$Y_r(x) \sim \mathcal{U}(B_r(x)).$$

Let γ be a uniform geodesic from x to $Y_n(x)$. For $i \in \{0, \ldots, r\}$ we define

$$Y_i(x) := \gamma_{i \wedge d(x, Y_r(x))}.$$

(i) Prove that for every $i \in \{0, \ldots, r\}$ and for every $y \in V$,

$$\sum_{x \in V} \mathbb{P}[Y_i(x) = y] = 1$$

(ii) Deduce that for every $i \in \{0, \ldots, r\}$ and for every $K \subset V$ finite,

$$\sum_{x \in V} \mathbb{P}[Y_i(x) \in \partial^{\text{in}} K] \leq |\partial^{\text{in}} K|.$$

(iii) Let $K \subset V$ finite. Prove that for every $i \in \{0, \ldots, r\}$,

$$\sum_{x \in K} \mathbb{P}[Y_i(x) \notin K] \leq r \cdot |\partial^{\text{in}} K|.$$

(iv) Prove that for $r = R(2|K|)$ and for every $i \in \{0, \ldots, r\}$,

$$\sum_{x \in K} \mathbb{P}[Y_i(x) \notin K] \geq \frac{|K|}{2}.$$

(v) Conclude the Theorem.