ETH Zürich, HS 2022 | Coordinator
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RANDOM WALKS ON TRANSITIVE GRAPHS (D-MATH)
EXERCISE SHEET 5

($\star$) Exercise 1. Let $Z \geq 0$ be a discrete random variable and $f : \mathbb{R} \to [0, \infty)$ a non decreasing function. Prove that

$$\mathbb{E}[Zf(Z)] > af(a), \text{ where } a = \frac{1}{2}\mathbb{E}[Z].$$

($\star$) Exercise 2. Let $X \geq 0$ be a discrete random variable such that $\mathbb{E}[X^2] = 1$. Consider Y the random variable defined by

$$\forall k \in \operatorname{supp}(X), \quad \mathbb{P}[y = k] = k^2 \mathbb{P}[X = k].$$

(i) Why is Y well defined?

(ii) Prove that for every function $g : \mathbb{R} \to [0, \infty)$, with $g(0) = 0$,

$$\mathbb{E}[g(X)] = \mathbb{E}\left[\frac{1}{Y^2} g(Y)\right].$$

(iii) Set $h : \mathbb{R} \to [0, \infty)$ be a non increasing function. Show that

$$\mathbb{E}[Xh(X)] \geq ah\left(\frac{1}{a}\right), \text{ with } a = \frac{1}{2}\mathbb{E}[X].$$

Exercise 3. Prove that $\forall t \in [0, 1]$,

$$\frac{\sqrt{1+t}}{2} + \frac{\sqrt{1-t}}{2} \leq 1 - \frac{t^2}{8}.$$