ETH Zürich, HS 2022 Coordinator
Prof. Vincent Tassion Daniel Contreras

RANDOM WALKS ON TRANSITIVE GRAPHS (D-MATH)
EXERCISE SHEET 6

In the following exercises we consider $G = (V, E)$ an infinite, locally finite, connected, and transitive graph of degree $d > 0$ with fixed origin $o \in V$. Let $p_n(o) := \mathbb{P}_o[X_n = o]$ be the return probability of the simple random walk (X_n) on G started at o.

($\star$) **Exercise 1.** Let $\rho(G)$ be the spectral radius of G.

(i) Prove that
$$\rho(G) = \lim_{n\to\infty} p_{2n}(o)^{\frac{1}{2n}}$$
and if G is not bipartite
$$\rho(G) = \lim_{n\to\infty} p_n(o)^{\frac{1}{n}}.$$

(ii) Prove that $\frac{1}{d} \leq \rho(G) \leq 1$.

($\star$) **Exercise 2.** Let P be the transition operator associated to the simple random walk on G and let $I : \mathcal{C} \to \mathcal{C}$ be the identity operator.

(i) Show that $Q := \frac{1}{2}(P + I)$ is the transition operator associated to the lazy random walk on G.

(ii) Calculate the spectral radius of the lazy random walk on G in terms of $\rho(G)$.

Definition. Let us consider the set of oriented edges $\vec{E} := \{(x, y) : \{x, y\} \in E\}$. For every $\theta, \psi : \vec{E} \to \mathbb{R}$ with finite support we define
$$\langle \theta, \psi \rangle_{\vec{E}} := \frac{1}{2} \sum_{e \in \vec{E}} \theta(e)\psi(e) \quad \text{and} \quad \|\theta\| := \sqrt{\langle \theta, \theta \rangle_{\vec{E}}}.$$
For every $f : V \to \mathbb{R}$, we define $\nabla f : \vec{E} \to \mathbb{R}$ by
$$\forall (x, y) \in \vec{E},\ (\nabla f)((x, y)) := f(y) - f(x).$$

($\star$) **Exercise 3.** Let P be the transition operator associated to the simple random walk on G and let $I : \mathcal{C} \to \mathcal{C}$ be the identity operator. Show that for every $f, g \in \mathcal{C}$
$$\langle (I - P)f, g \rangle = \frac{1}{d} \langle \nabla f, \nabla g \rangle_{\vec{E}}.$$

($\star$) **Exercise 4.** Let $\rho(G)$ be the spectral radius of G. Show that
$$\rho(G) = 1 - \frac{1}{d} \left(\inf_{f \in \mathcal{C} \setminus \{0\}} \frac{\|\nabla f\|}{\|f\|} \right)^2.$$