ETH Zürich, HS 2022 — Coordinator
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RANDOM WALKS ON TRANSITIVE GRAPHS (D-MATH)
EXERCISE SHEET 7

In the following exercises we consider $G = (V, E)$ an infinite, locally finite, connected, and transitive graph with fixed origin $o \in V$. Let (X_n) be the simple random walk on G started at o and for $x \in V$ we denote $p_n(o, x) := \mathbb{P}_o[X_n = x]$.

($\star$) Exercise 1.

(i) Assume there exist constants $A, C, \alpha > 0$ such that for all n, $|B_n| \leq Ae^{Cn^\alpha}$. Prove that

$$\limsup_{n\to\infty} \frac{|X_n|}{n^{1/(2-\alpha)}} \leq (2C)^{1/(2-\alpha)} \quad \mathbb{P}_o\text{-a.s.}$$

(ii) Assume there exist constants $A, D > 0$ such that for all n, $|B_n| \leq An^D$. Prove that

$$\limsup_{n\to\infty} \frac{|X_n|}{\sqrt{n \log n}} \leq \sqrt{D+1} \quad \mathbb{P}_o\text{-a.s.}$$

(iii) Assume that G has sub-exponential growth. Prove that

$$\frac{|X_n|}{n} \xrightarrow[n\to\infty]{} 0 \quad \mathbb{P}_o\text{-a.s.}$$

($\star$) Exercise 2.

(i) Let $A \subset V$ finite. Prove that

$$\mathbb{P}_o[X_n \in A] \leq 2\exp\left(-\frac{d(o,A)^2}{2n}\right)\sqrt{|A|},$$

where $d(o, A) = \min_{y\in A} d(o, y)$.

(ii) Deduce that

$$\min_{x\in\partial B_k} p_n(o, x) \leq \frac{2}{\sqrt{|\partial B_k|}} \exp\left(-\frac{k^2}{2n}\right).$$