ETH Zürich, HS 2022
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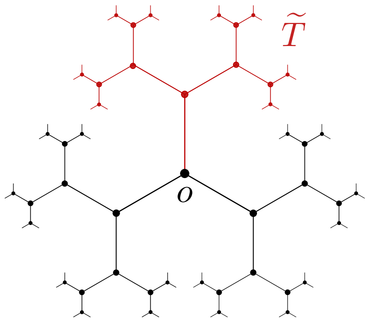
RANDOM WALKS ON TRANSITIVE GRAPHS (D-MATH)
EXERCISE SHEET 8

In the following exercises we consider $G = (V, E)$ an infinite, locally finite, connected, and transitive graph with fixed origin $o \in V$. Let (X_n) be the lazy random walk on G.

($\star$) **Exercise 1.** Consider $G = \mathbb{T}_3$ the 3-regular tree, and $\widetilde{T}$ a branch of this tree, as in the figure below. Prove that

$$h(x) := \mathbb{P}_x[(X_n) \text{ exits in } \widetilde{T}], \text{ for each } x \in V,$$

is harmonic and compute $h(x)$.



($\star$) **Exercise 2.** Let $D \subset V$ finite and connected. Let $T := \inf\{n : X_n \in D\}$. Show that

$$\forall x \in D,\ \mathbb{P}_x[T < \infty] = 1.$$

($\star$) **Exercise 3.** Let $x \in V$. Show that there exist (W_n), $(\widetilde{W_n})$ two lazy random walks starting from x such that

$$\mathbb{P}[\exists n_0 : \forall n \geq n_0,\ W_{n+1} = \widetilde{W_n}] = 1.$$

($\star$) **Exercise 4.** Let $\mathcal{T}$ be the tail σ-algebra of (X_n). We define

$$L^\infty(\mathcal{T}) := \{Z : \Omega \to \mathbb{R} : Z \text{ is } \mathcal{T}\text{-measurable, and } \exists C > 0, \mathbb{P}_o[|Z| \leq C] = 1\}.$$

Let $Z \in L^\infty(\mathcal{T})$. Show that $\forall x \in V,\ \forall n \geq d(o, x)$,

$$\mathbb{E}_o[Z \mid X_n = x] = \mathbb{E}_o[Z \mid X_{n+1} = x].$$