RANDOM WALKS ON TRANSITIVE GRAPHS_

Background:

- Probability theory. Applied stochastic processes (Markov chains).
- Group theory. (basic definitions → group given by generators and relations.)

General goal

Understand the relationship between the geometric properties of a graph and the behavior of a simple random walk on this graph.

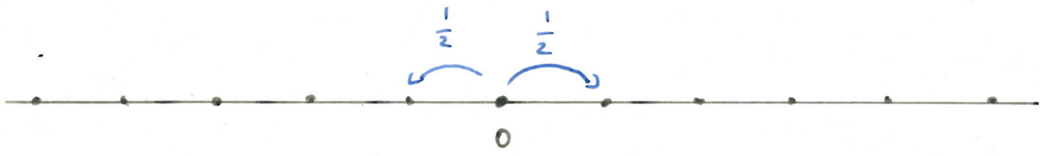
Framework

- $G = (V, E)$ infinite transitive graphs, $o \in V$ fixed origin.
  → informal def: "graph where all the vertices play the same role"

- (X_n) simple random walk on G, starting at o. (SRW)

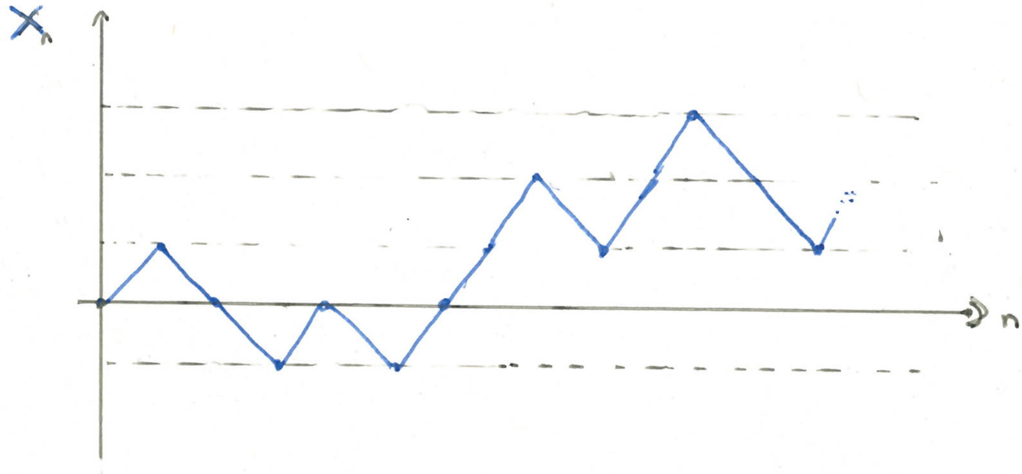
  ↳ $X_0 = o$
  X_{n+1} = uniformly chosen neighbour of X_n.

Ex: SRW on $\mathbb{Z}$



Let $(Z_i)_{i\geq 1}$ be iid r.v. with $\mathbb{P}[Z_i = -1] = \mathbb{P}[Z_i = +1] = \frac{1}{2}$.

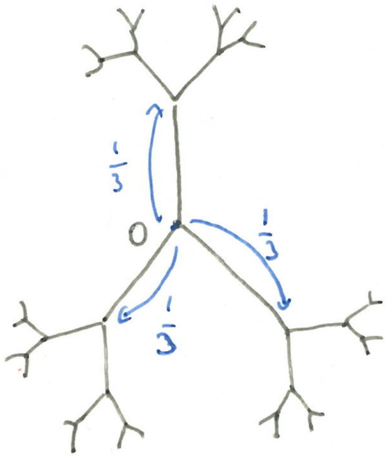
$X_n := \sum_{i=1}^{n} Z_i$ is a SRW on $\mathbb{Z}$.



position of the walker after n steps.

Ex: SRW on $\mathbb{Z}^d$, $d \geq 1$.

Ex: SRW on the 3-regular tree T_3.



QUESTION 1: RECURRENCE / TRANSIENCE ?

"Will the random walker visit the origin infinitely often?"

↳ on $\mathbb{Z}$, $\mathbb{Z}^2$: yes (the walk is recurrent)
↳ on $\mathbb{Z}^3$, $\mathbb{Z}^4$, ... : no (the walk is transient)

[POLYA '21]

↳ on T^3 : no (transient)

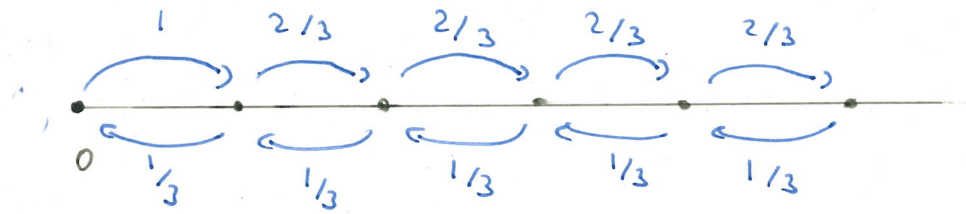
Idea of proof for the tree.

Let $(X_n)_{n \geq 0}$ be a SRW on $\mathbb{T}^3$. Then at each step. (except when $X_n = 0$)

→ jump away from 0 with proba $\frac{2}{3}$

→ jump towards 0 with proba $\frac{1}{3}$

One can prove that the distance $|X_n|$ of the walker from 0 is a Markov Chain with transition probabilities



Then, using the law of large numbers, one can show that

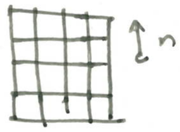
$$\frac{|X_n|}{n} \longrightarrow \frac{1}{3} \quad \text{a.s.}$$

and therefore, the walk is transient.

And for more general graphs?

Let $B_n = \{$vertices that can be reached by paths of length $\leq n\}$

Ex on $\mathbb{Z}$ $\quad B_n =$  $\quad |B_n| = 2n+1$

$\mathbb{Z}^d$ $\quad B_n =$  $\quad |B_n| = (2n+1)^d$

on $\mathbb{T}^3$ $\quad B_n =$  $\quad |B_n| = 1+3+6+\dots+3.2^{n-1}$

$= 3.2^n - 2$

We will prove that for general transitive graphs.

$$\big(\text{the SRW on } G \text{ is recurrent}\big) \iff \underbrace{\big(\exists C \text{ s.t. } \forall n\ |B_n| \leq C n^2\big)}$$

" volume growth at most quadratic "

↑ property of the walk

↑ geometric property of the graph.

QUESTION 2. WHAT IS THE RETURN PROBABILITY AFTER n STEPS?

$$p_n := \mathbb{P}[X_n = 0]$$ "return probability"

Thm (admitted)

on $\mathbb{Z}^d$: $\exists\, c = c(d) > 0$. s.t.

$$p_n \underset{n\to\infty}{\sim} \frac{c(d)}{n^{d/2}}$$ "poly. decay"

on $\mathbb{T}^3$; we have

$$p_n \leq \frac{1}{\sqrt{\pi n}} \left(\frac{8}{9}\right)^n (1+o(1))$$ "exponential decay"

And for more general graph?

Cheeger constant. $$\phi := \inf_{S \subset V} \frac{|\partial S|}{|S|}$$ "edge boundary"

where the infimum is over all connected subsets of V. and $\partial S = \{ \{x,y\} \in E,\ x \in S,\ y \notin S \}$

Ex: . for $\mathbb{Z}^d$ $\quad \Phi = 0 \quad$ (take $S = B_n$)

. for $\mathbb{T}^3$ $\quad \Phi = 1 \quad$ (Hint: Use that for every S

$$\sum_{x \in S} \underset{\substack{\uparrow \\ \text{in } \mathbb{T}^3}}{\deg(x)} = 2|E(S)| + |\partial S| \Big)$$

We will prove the following theorem, due to Kesten.

$$\left(\exists c>0 \text{ s.t } \forall n \ p_n \leqslant e^{-cn}\right) \Longleftrightarrow \left(\Phi > 0\right)$$

"property of the SRW" "geometric graph property"

QUESTION 3. SPEED OF THE SRW ?

$|X_n|$ = distance from 0 to X_n. What is

$$\rho := \lim_{n\to\infty} \frac{E[|X_n|]}{n} \quad ?$$

(existence of the limit? Is $\rho > 0$?)

on $\mathbb{Z}$: The random walk can be written

$$X_n = Z_1 + \cdots + Z_n$$

where $Z_1, \dots, Z_n$ iid uniform in $\{+1, -1\}$

Hence, the law of large numbers imply

$$\frac{X_n}{n} \xrightarrow[n\to\infty]{} E[Z_1] = 0$$

and therefore, by dominated convergence.

$$\frac{E[|X_n|]}{n} \xrightarrow[n\to\infty]{} 0 \qquad \text{"zero speed"}$$

Exercise: Prove that $\exists c > 0$ s.t.

$$\lim_{n\to\infty} \frac{E[|X_n|]}{\sqrt{n}} = c \qquad \text{"diffusive behaviour"}$$

on $\mathbb{Z}^d$, $d \geq 1$; equivalently.

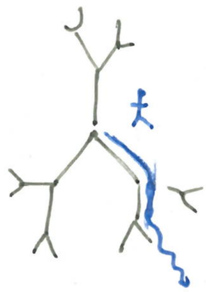
$$\frac{E[|X_n|]}{n} \longrightarrow 0$$

"zero speed"

on $\mathbb{T}^3$. We have seen

$$\frac{E[|X_n|]}{n} \xrightarrow[n\to\infty]{} \frac{1}{3}$$

"positive speed"



"the random walker escapes quickly to infinity"

What happens on more general graphs?

A function $h : V \longrightarrow \mathbb{R}$ is harmonic if

$$\forall x \in V \quad h(x) = \frac{1}{\deg(x)} \cdot \sum_{y \sim x} h(y)$$

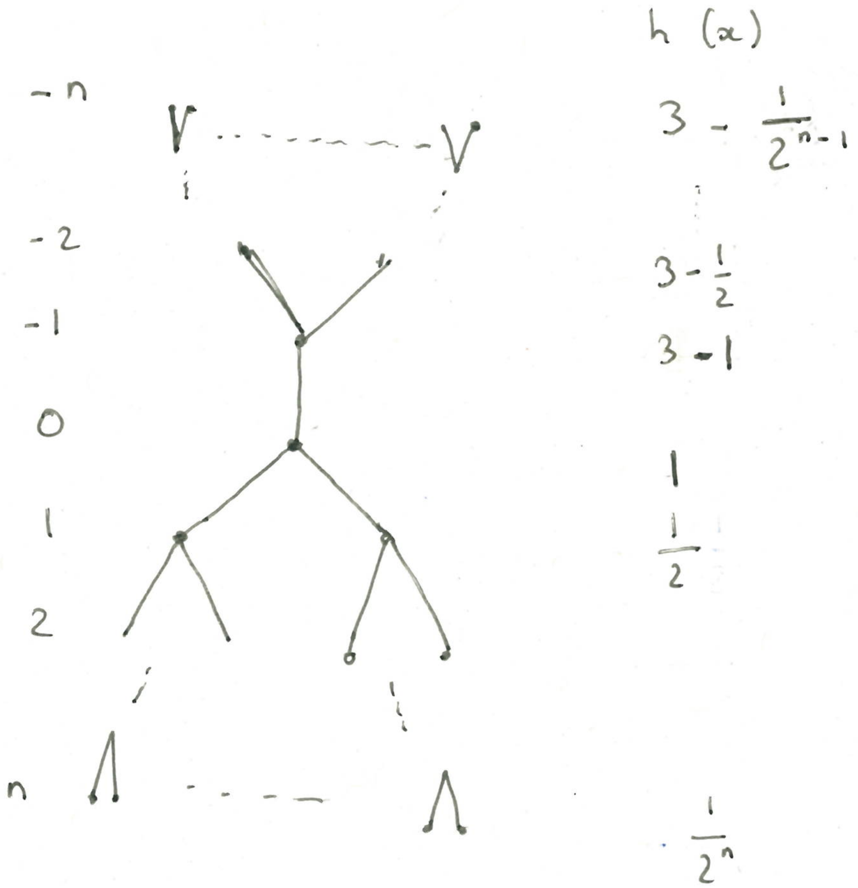
"neighbour"

Examples : $h = \text{cte}$ is always harmonic.

on $\mathbb{Z}$ $\quad h(x) = \lambda x + \mu$

on $\mathbb{Z}^2$ $\quad h(x,y) = xy$

on $\mathbb{T}^3$



We will prove that

$$\left(\ell \quad 0 \right) \Longleftrightarrow \left(\begin{array}{c} \text{the only bounded} \\ \text{harmonic functions} \\ \text{are constant} \end{array} \right)$$

"zero speed" "Liouville property"

CHAPTER 1 : TRANSITIVE GRAPHS

1) DEFINITIONS

Ref : [DIESTEL, Chap. 1] [GODSIL-ROYLE, Chap 1, 3]

Def: A graph is a pair $G = (V, E)$ where
- V is an arbitrary set
- E is a subset of $\{\{x,y\} : x, y \in V, x \neq y\}$

Notation. For $x, y \in V$ we write $xy := \{x, y\}$.
If $xy \in E$, we say that x and y are neighbours, and we write $x \sim y$.

Notice that the edges are unoriented $(xy = yx)$

Def: The degree of a vertex $x \in V$ is defined by

$$\deg(x) = \left|\{ y \in V : y \sim x \}\right| .$$

We say that G is locally finite if $\forall x \in V \ \deg(x) < \infty$.

Def: Two graphs $G = (V, E)$ and $G' = (V', E')$ are isomorphic if there exists a bijection $\phi : V \longrightarrow V'$ s.t.

$$(x \sim y \text{ in } G) \iff (\phi(x) \sim \phi(y) \text{ in } G')$$

In this case, we say that ϕ is an isomorphism from G to G'.

Ex

Diagrammatic representations of two isomorphic graphs

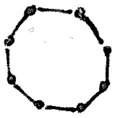
Rk: If ϕ isomorphism from G to G', then $\deg_G(x) = \deg_{G'}(\phi(x))$

$\hookrightarrow$  and  are not isomorphic.

Def: An automorphism of $G = (V, E)$ is an isomorphism from G to itself.

Not: $\mathrm{Aut}(G) = \{\text{automorphisms of } G\}$ group of automorphisms of G
(it is a subgroup of the group of the permutations of V)

Example:

$G = \mathbb{Z}/_{m}\mathbb{Z}$  $\mathrm{Aut}(G) = \{\tau^k, k = 0 \dots m-1\} \cup \{\tau^k \circ r, k = 0 \dots m-1\}$
where $\tau(x) = x+1$ and $r(x) = -x$
"diedral group of size $2m$".

Def: A graph $G = (V, E)$ is transitive iff

$$\boxed{\forall x, y \in V \quad \exists \phi \in \mathrm{Aut}(G) \text{ s.t. } \phi(x) = y}$$

Rk: Equivalently, G is transitive if the action of $\mathrm{Aut}(G)$ on V is transitive. (As a subgroup of the permutations of V, $\mathrm{Aut}(G)$ acts naturally on V : for $\phi \in \mathrm{Aut}(G)$, $x \in V$ $\phi \cdot x = \phi(x)$)

Rk: If G is transitive, $\deg(x)$ does not depend on $x \in V$ and is called the degree of G.

2. CAYLEY GRAPHS

Ref: [LYONS-PERES, Section 3.4] [SISTO] [DE LA HARPE, Chapters 2,4]

Def.: Let Γ be a group, let $S \subset \Gamma$
We say that
- S generates Γ if the smallest subgroup of Γ containing S is Γ.
- S is symmetric if $S^{-1} = S$, ie $s \in S \Leftrightarrow s^{-1} \in S$