CHAPTER 7

SPEED

Ref. [LYONS PERES, CHAP. 14] [YADIN, Chap. 7]

Setup. . $G=(V,E)$ transitive, loc. finite, connected, infinite graph. degree d, fixed origin $o \in V$.

. $(X_n)_{n\in\mathbb{N}}$ LRW from x under $\mathbb{P}_x$

goal: study of $\mathbb{E}_o[|X_n|]$, distance between o and X_n.

1 DEFINITION AND FIRST EXAMPLES.

Write $|x| = d(o,x)$.

We have, for every $m,n \geq 0$

$$\mathbb{E}_o[|X_{m+n}|] \leq \mathbb{E}_o[|X_m| + d(X_m, X_{m+n})]$$
$$= \mathbb{E}_o[|X_m|] + \underbrace{\mathbb{E}_o[d(X_m, X_{m+n})]}$$
$$= \mathbb{E}_o[|X_n|]$$

(by invariance + Markov property)

By Fekete's lemma, $\left(\frac{\mathbb{E}_o[|X_n|]}{n}\right)_{n\geq 0}$ converges in $[0,\infty)$.

def: The <u>speed of the LRW</u> is defined by

$$\ell = \lim_{n\to\infty} \frac{E_0[|X_n|]}{n} = \inf_{n\geq 0} \frac{E_0[|X_n|]}{n}.$$

Prop. (i) $\ell = 0$ if G a sub-exponential growth

(ii) $\ell > 0$ if G is non-amenable.

proof. (i) Let $\delta > 0$.

$$E_0[|X_n|] = \underbrace{E_0[|X_n| \mathbb{1}_{|X_n|\leq \delta n}]}_{\leq \delta n} + \underbrace{E_0[|X_n| \mathbb{1}_{|X_n|>\delta n}]}_{\overset{V.C}{\leq} 2 \sum_{k\geq \delta n} |B_k| e^{-\frac{k^2}{2n}}}$$

$$\leq 2 \sum_{k\geq \delta n} |B_k| e^{-\frac{\delta k}{2}}$$

$$\leq \underbrace{2 \sum_{k\geq 0} |B_k| e^{-\frac{\delta k}{2}}}_{= C(\delta) < \infty \text{ indep. of } n}$$

Hence $\limsup_{n\to\infty} \frac{E_0[|X_n|]}{n} \leq \delta$.

(ii) Let $\rho < 1$ be the spectral radius of the LRW on G.

Let $0 < a < \dfrac{\log(1/\rho)}{\log(d)}$

$$\mathbb{P}[X_n \in B_{an}] \leq |B_{an}| \, \rho^{n + o(n)}$$
$$\leq (d^a \ast \rho)^{n+o(n)} \xrightarrow[n \to \infty]{} 0 .$$

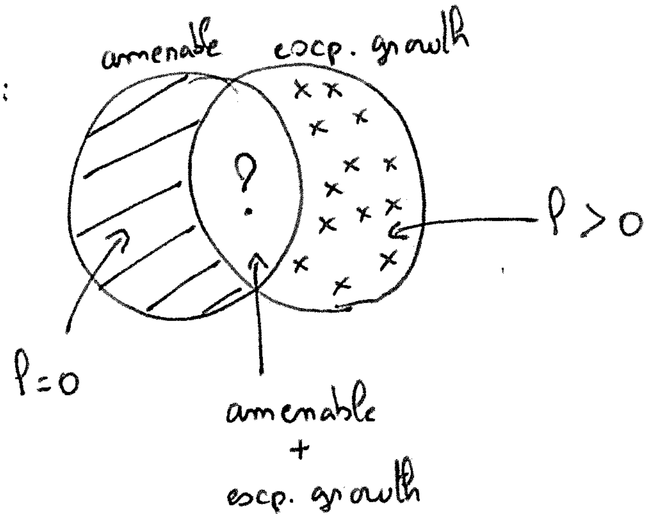
By Markov inequality,

$$\frac{\mathbb{E}[|X_n|]}{n} \geq a \ast \mathbb{P}[|X_n| \geq an]$$

By taking the limit as $n \to \infty$ and then taking the sup over $a \in \left(0, \frac{\log(1/\rho)}{\log(d)}\right)$, we obtain

$$\boxed{\ell \geq \frac{\log(1/\rho)}{\log d} > 0}$$

Rk:



Rk: We also have

$$\frac{|X_n|}{n} \xrightarrow[n\to\infty]{} \rho \quad P_0\text{-a.s.}$$

(see Thm 14.10 in [LYONS-PERES], follows from an application of Kingman's subadditive theorem).

2. ENTROPY AND SPEED

Th: $\boxed{\rho = 0 \iff h = 0}$

Pf: $\boxed{\Rightarrow}$ $H(X_n) = H(X_n, |X_n|)$ (because $|X_n|$ is X_n-measurable)

$$= \underbrace{H(|X_n|)}_{\leq \log n} + \underbrace{H(X_n \mid |X_n|)}$$

$$\overset{\text{Exercise}}{\leq} E_0\left[\log\left(|D_{|X_n|}|\right)\right]$$

$$\leq E_0\left[\log\left(d^{|X_n|}\right)\right]$$

$$\leq (\log d) \times E_0[|X_n|]$$

Dividing by n and taking the limit as $n \to \infty$, we obtain $h \leq (\log d)\,\rho$, which concludes the proof of the first implication.

$\boxed{\Leftarrow}$ For the second implication, we show $h \geq \frac{1}{2}\rho^2$.

By Varopoulos - Carne bound, we have

$$P_0[X_n = x] \leq 2e^{-\frac{|x|^2}{2n}}$$

Therefore,

$$H(X_n) = \sum_x \mathbb{P}_0[X_n = x] \log\left(\frac{1}{\mathbb{P}_0[X_n = x]}\right)$$

$$\geq \sum_x \mathbb{P}_0[X_n = x]\left(-\log 2 + \frac{|x|^2}{2n}\right)$$

$$= -\log 2 + \frac{1}{2n}\mathbb{E}[|X_n|^2]$$

$$\geq -\log 2 + \frac{1}{2n}\mathbb{E}[|X_n|]^2 \quad \text{(by Jensen's inequality)}$$

Dividing by n and taking $n \to \infty$, we obtain $h \geq \frac{1}{2}\ell^2$. ■

3 ENTROPY AND TAIL-TRIVIALITY.

$$\boxed{\underline{\text{Th}}:\quad h = 0 \iff \mathcal{T} \text{ trivial}}$$

$h = \overset{\text{see lemma 3 below}}{\text{increment}}$ of information in one step of the random walk.

$h = 0 \iff$ "X_1 does not bring information on X_n for n large"

Lemma 1 Let $n \geq 1$. For every Y $\overset{\text{real random variable}}{\text{r.r.v.}}$ meas. with respect to $X_0, X_1, \dots, X_n$, we have

$$\mathbb{E}_0[Y \mid X_n, X_{n+1}, \dots] = \mathbb{E}_0[Y \mid X_n] \quad \mathbb{P}_0\text{-a.s.}$$

Proof. Let $Z = f(X_n, X_{n+1}, \dots)$ be a bounded r.v. meas. wrt $X_n, X_{n+1}, \dots$

$$\mathbb{E}_0[YZ] = \mathbb{E}_0[\mathbb{E}_0[YZ \mid X_n]]$$

$$\overset{MP}{=} \mathbb{E}_0[\mathbb{E}_0[Y \mid X_n]\, g(X_n)] \qquad g(x) = \mathbb{E}_x[f(X_0, X_1, \dots)]$$

The same identity holds for $Y' = \mathbb{E}_0[Y \mid X_n]$ in place of Y.

Hence $\mathbb{E}_0[Y'Z] = \mathbb{E}_0[\, \mathbb{E}_0(Y' \mid X_n)\, g(X_n)]$

$$= \mathbb{E}_0[\mathbb{E}_0(Y \mid X_n]\, g(X_n)]$$
$$= \mathbb{E}_0[YZ]$$

Since Y' is $X_n, X_{n+1}, \dots$ measurable, we obtain

$$\mathbb{E}_0[Y \mid X_n, X_{n+1} \dots] = Y' \quad \mathbb{P}_0\text{-a.s.}$$ ∎

Lemma 2: Let $k \geq 0$ fixed integer. Let Y be a real r.v. meas. wrt $X_0, \dots, X_k$. Then

$$\lim_{n\to\infty} H(Y \mid X_n) = H(Y \mid \mathcal{T}).$$

Proof. Let $n \geq k$. By lemma 1, we have

$$H(Y \mid X_n) = H(Y \mid X_n, X_{n+1}, \dots)$$

By retrograde martingale convergence, we have

$$\lim_{n\to\infty} H(Y \mid X_n, X_{n+1}, \dots) = H(Y \mid \mathcal{T})$$ ∎

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Lemma 3.

$$\boxed{\lim_{n\to\infty} H(X_{n+1}) - H(X_n) = h}$$

Proof. For every n, let $u_n = H(X_n) - H(X_{n+1})$

We have $u_n \overset{MP}{=} H(X_n) - H(X_n | X_1)$

$$= H(X_n) + H(X_1) - H(X_1, X_n)$$

$$= H(X_1) - H(X_1 | X_n)$$

Hence by Lemma 2, $\lim_{n\to\infty} u_n$ exists.

By cesaro's theorem, we also have

$$\lim_{n\to\infty} \frac{u_1 + \cdots + u_n}{n} = \lim_{n\to\infty} u_n .$$

Since $\frac{u_1 + \cdots + u_n}{n} \xrightarrow[n\to\infty]{} h$ we deduce $\lim_{n\to\infty} u_n = h$ ∎

Proof of the Theorem

$$H(X_1) - H(X_1 | \mathcal{T}) \overset{\text{lem. 2}}{=} \lim_{n\to\infty} H(X_1) - H(X_1 | X_n)$$

$$= \lim_{n\to\infty} H(X_1) + H(X_n) - H(X_1, X_n)$$

$$= \lim_{n\to\infty} H(X_n) - \underbrace{H(X_n | X_1)}_{H(X_{n-1})}$$

$$\overset{\text{lem. 3}}{=} h$$

Equivalently, for every $k \geq 1$

$$H(X_1, \dots, X_k) - H(X_1, \dots X_k \mid \mathcal{T}) = k \cdot h.$$

$h = 0 \iff \forall k \geq 1 \quad X_1, \dots, X_k$ indep. of $\mathcal{T}$

$\iff \mathcal{T}$ is trivial.

↳ as in the proof of Kolmogorov 0-1 law. ∎

④ CONCLUSION

We have proved

Th: The following are equivalent:

(i) $\ell = 0$

(ii) $h = 0$

(iii) $\mathcal{T}$ trivial

(iv) G has LP.

Corollary: • If G has subexp. growth, G has LP (e.g. $\mathbb{Z}^d$)

• If G is non amenable, G does not have LP.