CHAP. 9

THE LAMPLIGHTER GRAPH.

Ref: [LYONS-PERES p. 88-89, p. 478] [WOESS 2000]

$G = (V, E)$ transitive locally-finite.

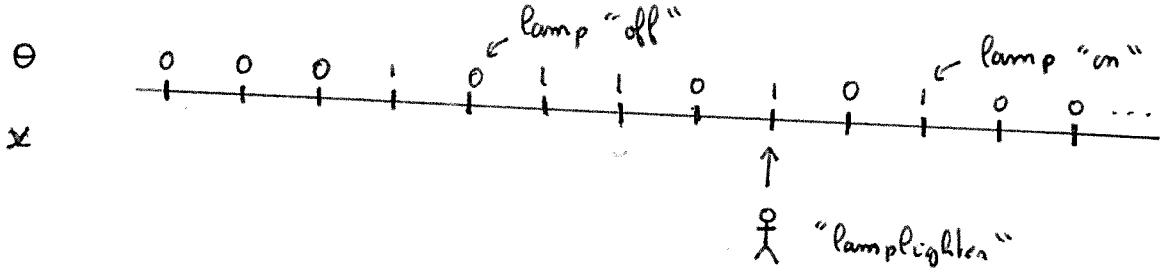
Not. Let $\mathcal{H}$ be the set of functions $\theta \in (\mathbb{Z}/2\mathbb{Z})^V$ with finite support (ie $\{x : \theta_x \neq 0\}$ is finite)

Def: $LL(G)$ is the graph with vertex set $\mathcal{H} \times V$ and edge set defined by

$$(\theta, x) \sim (\theta', x') \iff \begin{cases} x = x' & \text{and } \theta' = \theta + \mathbb{1}_{\{x\}} \\ x \sim x' & \text{and } \theta' = \theta \end{cases}$$

Geometric interpretation (for $G = \mathbb{Z}$)

(θ, x) vertex of $LL(G)$



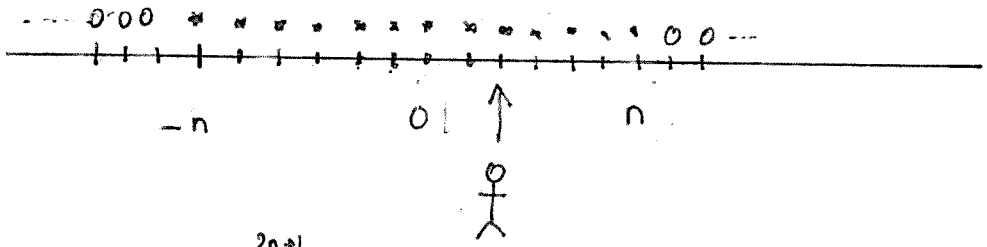
Possible moves: 1) the lamplighter switches on or turns off the lamp at its position.

2) the lamplighter moves to one of its neighbours.

Thm: (i) $LL(\mathbb{Z})$ has exponential volume growth.

(ii) $LL(\mathbb{Z})$ is amenable

Proof: Let $\mathcal{H}_n = \{\theta \in \mathcal{H} : \theta_x = 0 \text{ for } |x| > n\}$ and $S_n = \mathcal{H}_n \times \{-n, \dots, n\}$



We have $|S_n| = 2^{2n+1}(2n+1)$ and $S_n \subset B_{6n+1}$

Hence $|B_{6n+1}| \geq 4^n$.

This concludes (i).

Now $\partial S_n = \{\{(\theta,-n),(\theta,-n-1)\}, \theta \in \mathcal{H}_n\} \cup \{\{(\theta,n),(\theta,n+1)\}, \theta \in \mathcal{H}_n\}$

$$|\partial S_n| = 2 \times 2^{2n+1}$$

And therefore $\frac{|\partial S_n|}{|S_n|} = \frac{2}{2n+1} \xrightarrow[n\to\infty]{} 0$ which proves (ii) ∎

Rk1: The volume growth exponent of $LL(\mathbb{Z})$ can be computed exactly; Using that the set of vertices at distance exactly n from 0 "at the right of 0" is the Fibonacci sequence, one can show that $v = \log\left(\frac{1+\sqrt{5}}{2}\right)$. (see exercices)

Rk2: More generally if G is infinite $LL(G)$ has exponential volume growth and $(G \text{ amenable}) \Leftrightarrow (LL(G) \text{ amenable})$

Rk3 If (Γ, S) is a finitely generated group, there exists a group denoted $\Gamma \wr \mathbb{Z}/2\mathbb{Z}$ and a natural generating set $\tilde{S}$ s.t. $Cay(\Gamma \wr \mathbb{Z}/2\mathbb{Z}, \tilde{S}) = LL(Cay(\Gamma, S))$.

("$\wr$": "wedge product")

"Lamplighter group." see [LYONS PERES p.88]