**Def:** A group $\Gamma$ is <u>finitely generated</u> if $\exists S \subset \Gamma$ finite that generates $\Gamma$.

**Def:** Let $\Gamma$ be a finitely generated group. Let $S \subset \Gamma$ be a finite symetric set generating $\Gamma$.
The <u>Cayley graph</u> $Cay(\Gamma, S)$ associated to $(\Gamma, S)$ is the graph with
- vertex set $V = \Gamma$
- edge set $E = \{\{g, gs\} : g \in \Gamma, s \in S\} = \{gg' : g^{-1}g' \in S\}$

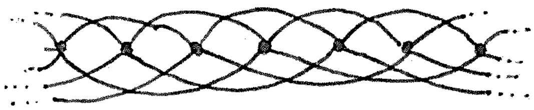
<u>Convention</u>: we will always assume that the neutral element of $\Gamma$ does not belong to $S$.

<u>Rk</u>: • $Cay(\Gamma, S)$ is transitive (exercice)

• The definition above corresponds to the right Cayley graph (in the definition of the edges $\{g, gs\}$, the generator "s" on the "right")
Alternatively, one could have defined the left-Cayley graph, which is isomorphic to the right Cayley graph, via $g \to g^{-1}$.

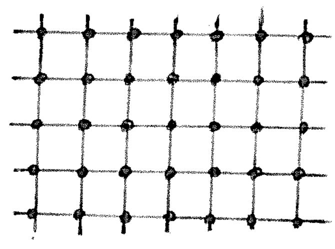
<u>Examples</u>

1) $\Gamma = \mathbb{Z}$ $S = \{-1, +1\}$ $Cay(\Gamma, S) =$ 

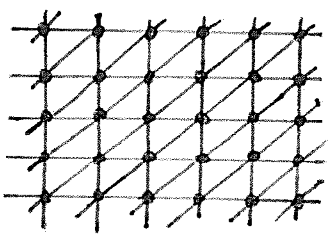
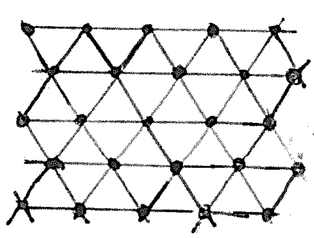
2) $\Gamma = \mathbb{Z}$ $S = \{\pm 2, \pm 3\}$ $Cay(\Gamma, S) =$ 

"changing the generating set gives rise to a different Cayley graph"

3) $\Gamma = \mathbb{Z}^2$ $S = \{\pm(0,1), \pm(1,0)\}$

$Cay(\Gamma, S) =$  "square lattice"

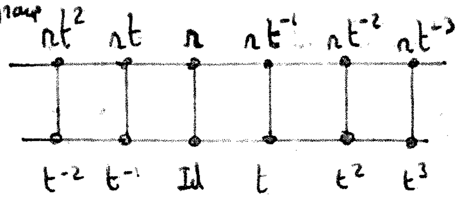
4) $\Gamma = \mathbb{Z}^2$ $S = \{\pm(0,1), \pm(1,0), \pm(1,1)\}$

$Cay(\Gamma, S) =$  $\cong$ isomorphic  "triangular lattice"

5) $\Gamma = \mathrm{Aut}(\mathbb{Z})$ $\quad S = \{t^{-1}, t, r\}$ where $t : \begin{array}{l} \mathbb{Z} \to \mathbb{Z} \\ x \mapsto x+1 \end{array}$ $\quad r : \begin{array}{l} \mathbb{Z} \to \mathbb{Z} \\ x \mapsto -x \end{array}$

"$\Gamma = D_\infty$ the infinite diedral group"

$\mathrm{Cay}(\Gamma, S) =$




"ladder graph"

(Notice $rt^{-k} = t^k r$)

Rk: Two different groups may have the same Cayley graph (see e.g. $D_\infty$ and $\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$)

## Free group over a finite set

Def: Let $A$ be a finite set. Let $S = A \cup \{a^{-1}, a \in A\}$ ($a^{-1}$ is just a formal symbol).

The set of words on $S$ is the set $W(S)$ of finite sequences of elements of $S$.

$W(S)$ is a monoid : the unit is the empty word, the product is given by the juxtaposition.

Let $\sim$ be the equivalence relation on $W(S)$ generated by

$$w s s^{-1} w' \sim w w' \quad \text{and} \quad w s^{-1} s w' \sim w w'$$

for every $s \in A$ and every $w, w' \in W(S)$.

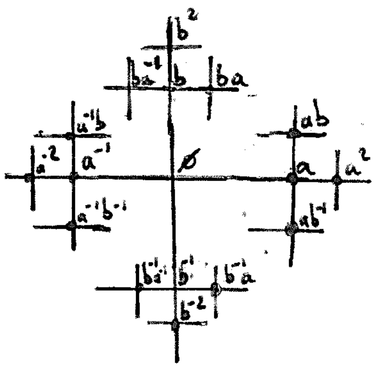
The Free group over $A$ is defined by $\mathbb{F}_A = W(S)/\sim$

Exple: $A = \{a\}$ $\quad S = \{a^{-1}, a\}$ $\quad \mathbb{F}_A = \{a^k, k \in \mathbb{Z}\}$

$\mathrm{Cay}(\mathbb{F}_A, S) =$


$A = \{a, b\}$ $\quad S = \{a^{-1}, a, b^{-1}, b\}$ $\quad$ ($\mathbb{F}_A = \mathbb{F}_2$ free group on two elements)

$\mathrm{Cay}(\mathbb{F}_2, S) =$




"4-regular tree"

## Groups defined by generators and relations

Let $S$ be a finite set and $R \subset \mathbb{F}_S$.
The group of presentation $\langle S \mid R \rangle$ is the quotient of $\mathbb{F}_S$ by the normal subgroup generated by $R$.

$S = \{$"generators"$\}$ $R = \{$"relators"$\}$

The presentation is said to be finite if $S$ and $R$ are finite.

Examples: $\mathbb{Z}^2 = \langle a,b \mid aba^{-1}b^{-1} \rangle$ $\left( \overset{\text{not.}}{=} \langle a,b \mid ab=ba \rangle \right)$

$\mathbb{Z}/_{n\mathbb{Z}} = \langle a \mid a^n \rangle$ $\left( \overset{\text{not.}}{=} \langle a \mid a^n = 1 \rangle \right)$

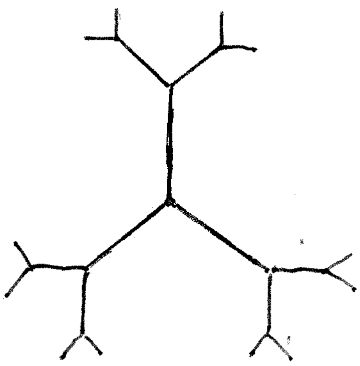
$D_{2n} = \langle t,r \mid t^n, r^2, rtrt \rangle$ $\left( \overset{\text{not.}}{=} \langle t,r \mid t^n=1, r^2=1, rtr^{-1}=t^{-1} \rangle \right)$

$\mathbb{F}_S = \langle S \mid \emptyset \rangle$

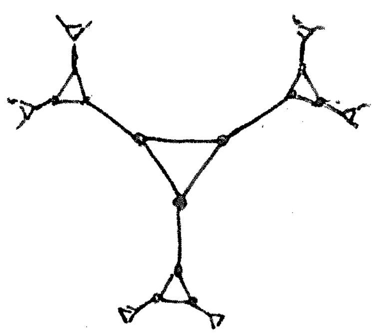
Def. The free product of $\Gamma_1 = \langle S_1 \mid R_1 \rangle$, $\Gamma_2 = \langle S_2 \mid R_2 \rangle, \dots, \Gamma_k = \langle S_k \mid R_k \rangle$ is the group $\Gamma_1 * \Gamma_2 * \dots * \Gamma_k$ of presentation $\langle S_1 \cup S_2 \cup \dots \cup S_k \mid R_1 \cup R_2 \cup \dots \cup R_k \rangle$

Rk: When a group is defined by a presentation $\langle S \mid R \rangle$ with $S$ finite, it is finitely generated (by the image of $S$ through the quotient).
In particular we can define its Cayley graph.

Ex:



$\mathbb{Z}/2\mathbb{Z} * \mathbb{Z}/_{2\mathbb{Z}} * \mathbb{Z}/_{2\mathbb{Z}}$
$(= \langle a,b,c \mid a^2, b^2, c^2 \rangle)$



$\mathbb{Z}/_{2\mathbb{Z}} * \mathbb{Z}/_{3\mathbb{Z}}$

## 3) METRIC STRUCTURE

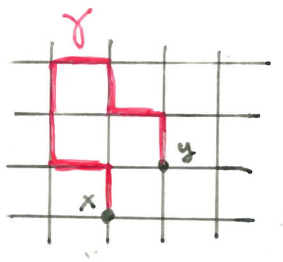
Ref [GODSIL - ROYLE, Chap 1]

Let $G=(V,E)$ be a locally finite transitive graph.

**Def:** A path of length $\ell$ from a vertex $x$ to a vertex $y$ is a sequence $\gamma=(\gamma_0,\gamma_1,\dots,\gamma_\ell)$ of distinct vertices s.t.

$$\gamma_0=x,\quad \gamma_\ell=y \quad \text{and } \forall i\ \ \gamma_{i-1}\sim\gamma_i .$$

**Def.** $G$ is said to be connected if for every $x,y\in V$, there exists a path from $x$ to $y$.




"A path from x to y"

**Def:** The distance between two vertices $x$ and $y$ is defined by

$$\boxed{d(x,y)=\min_{\gamma:x\to y}\left(\text{length}(\gamma)\right)}$$

where the minimum is over all the paths from $x$ to $y$.

**Rk:** The distance $d$ is invariant under the action of $\text{Aut}(G)$:

$$\boxed{\forall \phi\in \text{Aut}(G)\quad \forall x,y\in V\quad d(\varphi(x),\varphi(y))=d(x,y).}$$

**Pf:** Let $\gamma=(\gamma_0,\dots\gamma_\ell)$ be a path from $x$ to $y$ with $\ell=d(x,y)$.
Then $\varphi\cdot\gamma=(\varphi\cdot\gamma_0,\dots,\varphi\cdot\gamma_\ell)$ is a path from $\varphi(x)$ to $\varphi(y)$.
Hence $d(\varphi(x),\varphi(y))\le d(x,y)$. The reverse inequality is obtained by considering $\varphi^{-1}$. $\square$

# 4 GROWTH

Ref: [LYONS - PERES, p.472] [IMRICH - STEIFER]

$G=(V,E)$ infinite, locally-finite, transitive graph. $o \in V$ fixed origin.

**Def.** For $x \in V$, $n \geq 0$, <u>the ball of radius $n$ around $x$</u> is defined by

$$\boxed{B_n(x) = \{y \in V : d(x,y) \leq n\}}$$

Not: $B_n = B_n(o)$.

Rk: By transitivity, the graphs induced by $B_n(x)$ and $B_n(y)$ are isomorphic (exercice). In particular $|B_n(x)| = |B_n(y)|$

**Prop** (Definition of the <u>volume growth exponent</u>)

The following limit exists and is finite:

$$\boxed{v = \lim_{n\to\infty} \frac{1}{n} \log(|B_n|)} \quad \text{"volume growth exponent"}$$

In other words $|B_n| = e^{vn + o(n)}$

**Lem** (Fekete's subadditivity lemma.)

Let $(u_n)_{n\geq 0}$ be a sequence of numbers in $[-\infty, +\infty)$ satisfying

$$\forall m,n \geq 0 \quad u_{m+n} \leq u_m + u_n \qquad \text{"subadditivity"}$$

Then the limit of $\left(\frac{u_n}{n}\right)$ exists in $[-\infty, +\infty)$

$$\lim_{n\to\infty} \left(\frac{u_n}{n}\right) = \inf_{n\geq 0} \left(\frac{u_n}{n}\right)$$

Proof of the proposition:

We have $B_{m+n} = \bigcup_{x\in B_m} B_x(n)$. Hence $|B_{m+n}| \leq \sum_{x\in B_m} \underbrace{|B_n(x)|}_{=|B_n|}$

Therefore $|B_{m+n}| \leq |B_m| \cdot |B_n|$. By applying Fekete's lemma to $u_n = \log(|B_n|)$ we obtain $\frac{1}{n}\log(|B_n|) \longrightarrow \inf_n \left(\frac{\log(|B_n|)}{n}\right)$

($u_n$ is finite because $G$ is locally finite.) □

Rk: $\forall n \; |B_n| \geq e^{vn}$.

Rk: We have $\forall n \geq 1 \; |B_{n+1}| \leq (d-1)|B_n| + 2$ where $d$ is the degree of $G$. Hence $v \leq \log(d-1)$ and the bound is realized for the $d$-regular tree. (the relation $|B_{n+1}| \leq (d-1)|B_n| + 2$ can be proved by induction)

Def: We say that $G$ has

- exponential (volume) growth if $v > 0$
- polynomial (volume) growth if $\exists c < \infty$ s.t. $\forall n \; |B_n| \leq n^c$
- intermediate (volume) growth if $v = 0$ and $\forall c < \infty \; \sup_n \left(\frac{|B_n|}{n^c}\right) = +\infty$

Examples:
- $T_d$, $d \geq 3$ has exponential growth
- $\mathbb{Z}^d$, $d \geq 1$ has polynomial growth.
- there exists Cayley graphs of intermediate growth (Grigorchuck groups)

Thm: [Gromov, Trovimov]

If $G$ has polynomial volume growth, then $\underline{\underline{\exists k \in \mathbb{N}}}$ $\exists c_1, c_2 > 0$ s.t.

$$\forall n \geq 0 \quad c_1 n^k \leq |B_n| \leq c_2 n^k.$$

- This deep theorem was proved by Gromov in the framework of groups, he showed that a Cayley has polynomial volume growth if and only if the underlying group is virtually nilpotent.
- It was later extended to transitive graphs by Trovimov.

# CHAPTER 2

## – DEFINITIONS AND FIRST PROPERTIES –

Setup: $G = (V, E)$ transitive, locally-finite, connected, infinite graph degree $d$, fixed origin $o \in V$.

## 1 DEFINITION

Def: Let $x \in V$. A simple random walk (SRW) on $G$ starting at $x$ is a sequence of random variables $(X_n)_{n\in\mathbb{N}}$ with values in $V$ such that

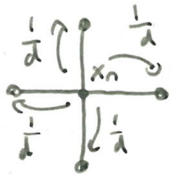
$\forall n \geq 0 \ \forall x_0, x_1, \dots, x_n \in V$

$$\boxed{\mathbb{P}[X_0 = x_0, \dots, X_n = x_n] = \mathbb{1}_{x_0 = x}\, p(x_0, x_1) \dots p(x_{n-1}, x_n)}$$

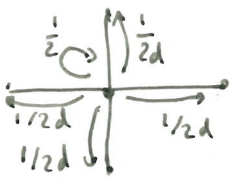
$$\boxed{\text{where } p(x,y) := \begin{cases} \frac{1}{d} & \text{if } x \sim y \\ 0 & \text{otherwise} \end{cases}}$$ "transition probabilities"

Def. Equivalently we define the lazy random walk (LRW) by considering the transition probabilities

$$\boxed{p(x,y) := \begin{cases} \frac{1}{2d} & x \sim y \\ \frac{1}{2} & x = y \\ 0 & \text{otherwise} \end{cases}}$$




SRW




LRW

Remarks

- A walk from $x$ of length $n$ is a sequence $\gamma = (\gamma_0, \dots, \gamma_n)$ such that $\gamma_0 = x$ and $\forall i \ \gamma_{i-1} \sim \gamma_i$ (the vertices are not necessarily disjoint).

$$\mathbb{P}[(X_0, \dots, X_n) = (x_0, \dots, x_n)] = \frac{1}{d^n} \mathbb{1}_{\left\{ \substack{(x_0, \dots, x_n) \text{ walk from } x \\ \text{of length } n} \right\}}$$

→ "$(X_0, \dots, X_n)$ is a uniformly chosen walk from $x$ of length $n$."

- A SRW (resp. LRW) starting at $x$ is a homogeneous Markov chain with
  - → state space $V$
  - → initial distribution $\delta_x$
  - → transition probabilities $(p(x,y))_{x,y \in V}$.

- Assume $G = (V, E) = \mathrm{Cay}(\Gamma, \underset{\text{finite sym.}}{S})$. Let $Z_1, Z_2, \dots$ be an i.i.d sequence of uniform random variables in $S$.
Then, $(X_n)$ defined by

$$\forall n \qquad X_n := x \cdot Z_1 \cdot Z_2 \cdots Z_n$$

is a SRW on $G$, starting at $x$.

<u>Setup.</u> $\cdot (\Omega, \mathcal{F})$ measurable, equipped with $(\mathbb{P}_x)_{x \in V}$.

$\cdot (X_n)_{n \in \mathbb{N}}$ r.v.s on $\Omega$ s.t., under $P_x$,

$(X_n)_{n \in \mathbb{N}}$ is a SRW (or a LRW) starting at $x$.

## 2 (SIMPLE) MARKOV PROPERTY.

$\mathcal{F}_n := \sigma(X_0, \dots, X_n)$

<u>Prop</u> (MP) Let $x, y \in V$, $k \in \mathbb{N}$. For every $f : V^{\mathbb{N}} \to \mathbb{R}$ measurable bounded, for every $Z$ $\mathcal{F}_k$-meas. bounded

$$\boxed{E_x\left[f((X_{k+n})_{n \geq 0}) \times Z \mid X_k = y\right] = E_y\left[f((X_n)_{n \geq 0})\right] E_x\left[Z \mid X_k = y\right]}$$

"Condition on $X_k = y$, $(X_{k+n})_{n \geq 0}$ is a SRW (or a LRW) starting at $y$, independent of $X_0, \dots, X_k$."

<u>Not:</u> $p_n(x, y) := \mathbb{P}_x[X_n = y]$

$p_n(x) := p_n(o, x)$

<u>Corollary:</u> $\forall m, n \geq 0$ $\forall x, y \in V$

$$\boxed{p_{m+n}(x, y) = \sum_z p_m(x, z)\, p_n(z; y)}$$ Chapman Kolmogorov

$\to$ "matrix product interpretation"