3. IRREDUCIBILITY

Prop: The SRW is an irreducible Markov Chain:

$$\forall x, y \in V \quad \exists m \geq 0 : \quad p_m(x,y) > 0$$

Proof: Since $G = (V, E)$ is connected, there exists a path $\gamma = (\gamma_0, \dots, \gamma_m)$ from x to y.

$$p_m(x,y) = \mathbb{P}_x [X_m = y]$$

$$\geq \mathbb{P}_x [X_0 = \gamma_0, \dots, X_m = \gamma_m]$$

$$= \frac{1}{d^m} > 0 \qquad \blacksquare$$

4 APERIODICITY.

Recall that the period of (X_n) is defined as

$$\gcd \{ n \geq 0 : p_n(0) > 0 \}$$

For the LRW, we have $p(0,0) = \frac{1}{2} > 0$. Hence, the LRW is always aperiodic.

For the SRW, we have $\forall n \quad p_{2n}(0) > 0$. Hence, the period of the SRW is either 1 (aperiodic case) or 2.

Prop. The following are equivalent:

(i) The SRW is aperiodic

(ii) G is not bipartite*.

Proof. (exercise)

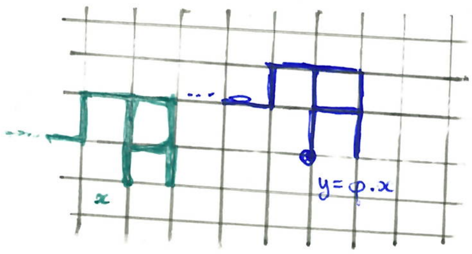
* G is said to be bipartite if $V = U \cup V$, where $U \cap V = \emptyset$ and every edge connects a vertex of U to a vertex of V.

5 INVARIANCE

Propn: $\boxed{\forall \varphi \in \mathrm{Aut}(G) \quad p(\varphi\cdot x, \varphi\cdot y) = p(x,y)}$

Consequence: • $\forall n \quad p_n(\varphi\cdot x, \varphi\cdot y) = p_n(x,y)$

• Let $\varphi \in \mathrm{Aut}(G)$ be such that $\varphi\cdot x = y$. If $(X_n)_{n\geq 0}$ is a SRW starting at x, then $(\varphi\cdot X_n)_{n\geq 0}$ is a SRW starting at y.



"in $\mathbb{Z}^2$, the translate of a SRW from x is a SRW from y"

6 REVERSIBILITY

Propn: $\boxed{\forall x,y \in V \quad p(x,y) = p(y,x)}$

"the measure μ on V defined by $\mu_x = 1 \ \forall x \in V$ is reversible for the SRW"

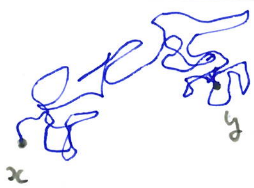
We say that the SRW is reversible (see [Aldous - Fill, chapter 3])

Consequences / geometric interpretation

• $\forall n \geq 0 \quad \forall x,y \in V \quad p_n(x,y) = p_n(y,x)$

• $\forall (\overset{x}{\overset{\shortparallel}{x_0}}, x_1, \dots, \overset{y}{\overset{\shortparallel}{x_n}}) \in V^{n+1}$

$$\mathbb{P}_x[(X_0,\dots,X_n) = (x_0,\dots,x_n)] = \mathbb{P}_y[(X_0,\dots,X_n) = (x_n,\dots,x_0)]$$



"the law of a n-step walk from x to y is that of a n-step walk from y to x."

Application:

$$\boxed{\forall m \geq 0 \ \forall x \in V \text{ we have } p_{2m}(0,0) \geq p_{2m}(0,x)}$$

Proof:

$$p_{2m}(0,0) = \sum_{y\in V} p_m(0,y)\, p_m(y,0) \qquad \text{(Chapman - Kolmogorov)}$$

$$= \sum_{y\in V} p_m(0,y)^2 \qquad \text{(Reversibility)}$$

$$= \sqrt{\sum_{y\in V} p_m(0,y)^2}\ \sqrt{\sum_{y\in V} p_m(x,y)^2} \qquad \text{(Invariance + transitivity of } G\text{)}$$

$$\geq \sum_{y\in V} p_m(0,y)\, p_m(x,y) \qquad \text{(Cauchy - Schwarz)}$$

$$= p_{2m}(0,x) \qquad \text{(Chapman - Kolmogorov + Reversibility)} \quad \blacksquare$$

CHAPTER 3 :

RETURN PROBABILITY.

Setup: • $G = (V, E)$ infinite, loc finite, connected transitive graph degree d, fixed origin $o \in V$.

• $(\Omega, \mathcal{F}, \mathbb{P}_x)_{x \in V}$ proba spaces and $(X_n)_{n \in \mathbb{N}}$ stoch. process s.t. $(X_n)_{n \in \mathbb{N}}$ LRW under $\mathbb{P}_x$. $\left(p(x,y) = \frac{1}{2} \mathbb{1}_{x=y} + \frac{1}{2d} \mathbb{1}_{x \sim y} \right)$

In this chapter, our goal is to provide general upper bounds on the return probability $p_n(o)$ of the LRW. These upper bounds will be expressed in terms of isoperimetric quantities of the graphs (which relate the boundary size and the volume of finite subsets).

In the first two sections we introduce the necessary geometric background. Then we define "evolving random sets" related to the random walk. By applying ~~these~~ isoperimetric bounds to these evolving sets, we will be able to obtain the desired upper bounds.

$$p_{2n}(o) \leq f(\text{"isoperimetric quantities"})$$

I ISOPERIMETRIC CONSTANT.

Ref: [LYONS-PERES, Chapter 6] [PATERSON, Introduction]

$G = (V,E)$ infinite locally-finite, transitive graph, $\deg(G) = d$.

Not: If $S \subset V$ finite, we write $\partial S = \{ xy : x \in S, y \in V \setminus S \}$

Def: The isoperimetric constant of G is defined by

$$\boxed{\Phi = \inf_{\substack{S \subset V \\ \text{finite}}} \frac{|\partial S|}{|S|}}$$

We say that G is
- amenable if $\phi = 0$
- non amenable if $\phi > 0$.

Rk: There exist many characterisations of amenability, see [PATERSON].
The definition originates from group theory.
A group Γ is amenable if there exist an invariant mean on Γ, ie a positive linear map $\Lambda \subset \mathrm{Hom}(L^\infty(\Gamma), \mathbb{R})$ of norm 1 and satisfying $\forall x \in \Gamma$ $\forall f \in L^\infty(\Gamma)$ $\Lambda(x \cdot f) = \Lambda(f)$.
In the context of finitely generated groups the definitions coincide.

Prop: If G is non amenable, then it has exponential growth

Proof: $$|B_{n+1}| = |B_n| + \underbrace{|\{ x \in V : d(x,0) = n+1 \}|}_{\geq \frac{1}{d} \cdot |\partial B_n|}$$

$$\geq \left(1 + \frac{\phi}{d}\right) |B_n|$$

By induction $|B_n| \geq \left(1 + \frac{\phi}{d}\right)^n$ $\square$

2 EXPANSION PROFILE.

If a graph is amenable, there exists a sequence of set $(S_n)_{n\in\mathbb{N}}$ s.t.

$$|\partial S_n| \ll |S_n|$$

How small can $|\partial S_n|$ be with respect to $|S_n|$?

→ on $\mathbb{Z}$, by choosing $S_n = \{-n, \dots, n\}$, we have

$$|\partial S_n| = 2 \qquad |S_n| = 2n+1$$

→ very small boundary. $|\partial S| \simeq$ cte

→ on $\mathbb{Z}^2$, the ball B_n satisfies $|\partial B_n| \simeq \sqrt{n}$ $\quad |B_n| \simeq n$

and we cannot do better : $\forall S \subset \mathbb{Z}^2$ finite $\quad |\partial S| \geq c|S|^{\frac{1}{2}}$.

In order to quantify the relation between $|\partial S|$ and $|S|$ we introduce the expansion profile.

Def. The expansion profile of G is the function φ defined by

$$\forall u \geq 1 \qquad \varphi(u) = \min_{1 \leq |S| \leq u} \left(\frac{|\partial S|}{|S|} \right)$$

Rk φ is ↘ and $\lim_{u\to\infty} \varphi(u) = \Phi$ isop. constant.

In particular G amenable $\iff \lim_{\infty} \varphi = 0$

Thm: Let $R(m) := \min\{ r \in \mathbb{N} : |B_r| \geq m \}$. There exists a constant $c > 0$ s.t.

$$\forall u \geq 1 \qquad \boxed{\varphi(u) \geq \frac{c}{R(2u)}}$$