Since $p(S_n, x) = \mathbb{P}[U_{n+1} \leq p(S_n, x) \mid S_n]$

$$= \mathbb{P}[x \in S_{n+1} \mid S_n],$$

we finally get $p_{n+1}(0,x) = \mathbb{P}[x \in S_{n+1}]$

(ii)
$$E[\,|S_{n+1}| \mid S_n] = \sum_{y \in V} \mathbb{P}[y \in S_{n+1} \mid S_n]$$

$$= \sum_{y \in V} \sum_{x \in S_n} p(x,y)$$

$$= \sum_{x \in S_n} \underbrace{\sum_{y \in V} p(x,y)}_{=1} = |S_n|$$

4 UPPER BOUND ON THE RETURN PROBABILITY

In this section we prove the following theorem.

Theorem:

Let φ be the expansion profile of G. Define $\varepsilon_n > 0$ by

$$\int_4^{4/\varepsilon_n} \frac{8d^2 \cdot du}{u\, \varphi^2(u)} = n .$$

Then we have $\boxed{p_{2n}(0,0) \leq \varepsilon_n}$.

Application

- $|B_n| \geq c\, n^d$ $(c>0)$ $\Rightarrow$ $\exists C>0$ $\quad p_{2n}(0) \leq \dfrac{C}{n^{d/2}}$

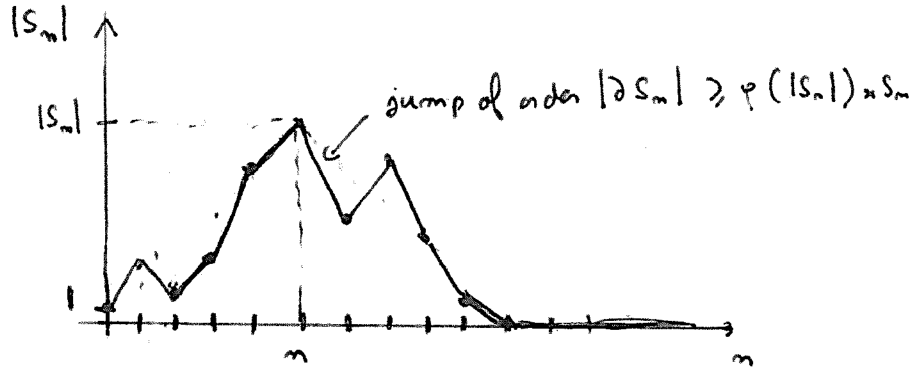
- $|B_n| \geq e^{cn}$ $(c>0)$ $\Rightarrow$ $\exists C>0$ $\quad p_{2n}(0) \leq C\, e^{-\frac{1}{C} n^{\frac{1}{3}}}$.

Rk: ε_n is well defined.

Idea of proof: We have $p_n(o) \leq \mathbb{P}[|S_n|=0]$

We want to bound $\mathbb{P}[|S_n|=0]$ where $|S_n|$ is martingale satisfying $|S_{n+1}| = |S_n| + Z_{n+1}$ where Z_n is a centered r.v. of order $\sqrt{E[Z_{n+1}^2]} \asymp |\partial S_n| \geq \varphi(|S_n|) \times S_n$. Roughly, $|S_n|$ is "faster" than the martingale M_n defined by $\begin{cases} M_0 = 1 \\ M_{n+1} = M_n + \varepsilon_{n+1}\, \phi(M_n) M_n \end{cases}$

where ε_n iid $\mathbb{P}(\varepsilon_i = 1) = \mathbb{P}[\varepsilon_i = -1) = \frac{1}{2}$.



↳ "reduction to a 1-D martingale problem". In order to get an estimate at which speed S_n gets absorbed at 0, we rather consider the supermartingale $\sqrt{|S_n|}$ which is "drifted" towards 0. ($\sqrt{|S_n|}$ is a super-martingale by Jensen inequality).

Introduce $\delta_n := E\left[\sqrt{|S_n|}\right]$.

Rk: $\delta_n \leq E\left[|S_n|\right]^{\frac{1}{2}} = 1$.

Lemma 1: For all $n \geq 0$ we have

$$\boxed{p_{2n}(0,0) \leq \delta_n^2}$$

Proof: Using Chapman-Kolmogorov for the lazy RW and reversibility, we find

$$p_{2n}(0,0) = \sum_{x \in V} p_n(0,x)^2 = \sum_{x \in V} \mathbb{P}[x \in S_n]^2$$

Introduce Σ_n, an independent copy of S_n.

$$p_{2n}(0,0) = \sum_{x\in V} \mathbb{P}[x\in S_n, x\in\Sigma_n]$$

$$= \mathbb{E}\Big[\sum_{x\in V} 1\!\!1_{x\in S_n} 1\!\!1_{x\in\Sigma_n}\Big]$$

$$= \mathbb{E}\big[|S_n\cap\Sigma_n|\big]$$

$$\leqslant \mathbb{E}\big[|S_n|\wedge|\Sigma_n|\big]$$

$$\leqslant \mathbb{E}\Big[\sqrt{|S_n||\Sigma_n|}\Big] = \delta_n^2 \quad \blacksquare$$

<u>Lemma 2.</u>

For every S s.t. $\mathbb{P}[S_n=S]>0$ and $S\neq\emptyset$, we have

- $\mathbb{E}\big[|S_{n+1}|\,\big|\,S_n=S, U_{n+1}<\frac{1}{2}\big] = |S| + \frac{1}{d}|\partial S|$, and
- $\mathbb{E}\big[|S_{n+1}|\,\big|\,S_n=S, U_{n+1}\geqslant\frac{1}{2}\big] = |S| - \frac{1}{d}|\partial S|$.

<u>Proof</u>: Since $(|S_n|)$ is a martingale, the second item follows from the first one.

For $y\in V$ we have

$$\mathbb{P}[y\in S_{n+1}\,|\,S_n=S, U_{n+1}<\tfrac{1}{2}] = \mathbb{P}[U_{n+1}\leqslant p(S,y)\,|\,U_{n+1}<\tfrac{1}{2}]$$

$$= \begin{cases} 1 & \text{if } y\in S \quad (\text{by laziness}) \\ \underbrace{2p(S,y)}_{=\frac{1}{d}\sum_{x\in S} 1\!\!1_{x\sim y}} & \text{if } y\notin S \end{cases}$$

Therefore,

$$E\left[|S_{n+1}| \,\middle|\, S_n = S, U_{n+1} < \tfrac{1}{2}\right] = |S| + \frac{1}{d} \sum_{y \notin S} \sum_{x \in S} 1\!\!1_{x \sim y} = |S| + \frac{1}{d} |\partial S| \quad \blacksquare$$

Lemma 3: For every $n \geq 0$, we have.

$$E\left[\sqrt{|S_{n+1}|} \,\middle|\, S_n\right] \leq \left(1 - \psi(|S_n|)\right) \sqrt{|S_n|}, \text{ where } \psi(u) := \frac{1}{8d^2} \varphi(u)^2.$$

Proof: We will use the following elementary inequality.

$$\forall t \in [0,1] \quad \frac{\sqrt{1+t}}{2} + \frac{\sqrt{1-t}}{2} \leq 1 - \frac{t^2}{8}. \text{ (exercice)}.$$

Let $S \subset V$ s.t. $\mathbb{P}[S_n = S] > 0$ and $S \neq \emptyset$

$$E\left[\sqrt{|S_{n+1}|} \,\middle|\, S_n = S\right] = \frac{1}{2} E\left[\sqrt{|S_{n+1}|} \,\middle|\, S_n = S, U_{n+1} < \tfrac{1}{2}\right] + \frac{1}{2} E\left[\sqrt{|S_{n+1}|} \,\middle|\, S_n = S, U_{n+1} \geq \tfrac{1}{2}\right]$$

Jensen + Lemma 2

$$\leq \frac{1}{2} \sqrt{|S| + \frac{1}{d} |\partial S|} + \frac{1}{2} \sqrt{|S| - \frac{1}{d} |\partial S|}$$

$$= \sqrt{|S|} \left(\frac{1}{2} \sqrt{1 + \frac{1}{d} \frac{|\partial S|}{|S|}} + \frac{1}{2} \sqrt{1 - \frac{1}{d} \frac{|\partial S|}{|S|}} \right)$$

$$\leq \sqrt{|S|} \left(1 - \frac{1}{8d^2} \left(\frac{|\partial S|}{|S|} \right)^2 \right)$$

$$\leq \sqrt{|S|} \Big(1 - \underbrace{\frac{1}{8d^2} \varphi(|S|)^2}_{= \psi(|S|)} \Big) \quad \blacksquare$$

Proof of the theorem.

$$\delta_{n+1} = E\left[\sqrt{|S_{n+1}|}\right] \underset{\text{Lemma 3}}{\leq} E\left[\sqrt{|S_n|}\,\left(1-\psi(|S_n|)\right)\right] = \delta_n - E\Big[\underbrace{\sqrt{|S_n|}}_{\nearrow \text{ in } |S_n|}\ \underbrace{\psi(|S_n|)}_{\searrow \text{ in } |S_n|}\Big]$$

Observation: $1 = E[|S_n|] = \sum_{S \subset V} |S|\, \mathbb{P}[S_n = S]$

We introduce a random variable $S_n^\bullet$ which is a size-biased version of S_n, define by

$$\mathbb{P}[S_n^\bullet = S] = |S|\, P[S_n = S]$$

This way, we have for every f measurable

$$E[f(S_n)\cdot|S_n|] = E[f(S_n^\bullet)].$$

Using this new variable, we obtain $\delta_n = E\left[\frac{1}{\sqrt{|S_n^\bullet|}}\right]$

$$E\left[\sqrt{|S_n|}\ \psi(|S_n|)\right] = E\left[\frac{1}{\sqrt{|S_n^\bullet|}}\ \psi(|S_n^\bullet|)\right]$$

$$\geq E\Big[\frac{1}{\sqrt{|S_n^\bullet|}}\ \psi(\underbrace{|S_n^\bullet|}_{\leq \frac{4}{\delta_n^2}})\ \mathbb{1}_{\frac{1}{\sqrt{|S_n^\bullet|}} \geq \frac{\delta_n}{2}}\Big]$$

$$\geq \psi\left(\frac{4}{\delta_n^2}\right) E\left[\frac{1}{\sqrt{|S_n^\bullet|}}\ \mathbb{1}_{\frac{1}{\sqrt{|S_n^\bullet|}} \geq \frac{\delta_n}{2}}\right]$$

$$\geq \psi\left(\frac{4}{\delta_n^2}\right)\cdot \frac{1}{2}\,\delta_n .$$

For the last inequality we use that for a positive integrable random variable X, $E[X\,\mathbb{1}_{X \geq \frac{1}{2}E[X]}] = E[X] - E[X\,\mathbb{1}_{X < \frac{1}{2}E[X]}] \geq \frac{1}{2} E[X]$.

Exercise. (see exercise sheet for intermediate questions)

Let $X \geq 0$ be a discrete r.v. with $\mathbb{E}[X^2] = 1$.

Let $f : \mathbb{R} \to \mathbb{R}_+$ be a nonincreasing function.

Prove that

$$\boxed{\mathbb{E}[X f(X)] \geq a \cdot f\left(\tfrac{1}{a}\right) \quad \text{where } a := \tfrac{1}{2}\mathbb{E}[X]}$$

Proof of the theorem.

Let $n \geq 0$. By Lemma 3, we have

$$\delta_{n+1} = \mathbb{E}[\sqrt{|S_{n+1}|}] \leq \mathbb{E}[\sqrt{|S_n|} \times (1 - \Psi(|S_n|))]$$

$$= \delta_n - \mathbb{E}[\sqrt{|S_n|}\, \Psi(|S_n|)]$$

$$\underset{\substack{\text{Exercise} \\ \text{with } X = \sqrt{|S_n|}}}{\leq} \delta_n - \frac{\delta_n}{2} \Psi\left(\frac{4}{\delta_n^2}\right).$$

Finally, we obtain $\forall m \geq 0$

$$\delta_{m+1} \leq \delta_m \left(1 - \frac{1}{2}\psi\left(\frac{4}{\delta_m^2}\right)\right).$$

This implies

$$\frac{\delta_{m+1}}{\delta_m} \leq e^{-\frac{1}{2}\psi\left(\frac{4}{\delta_m^2}\right)}$$

ie $$-\log(\delta_{m+1}) + \log(\delta_m) \geq \frac{1}{2}\psi\left(\frac{4}{\delta_m^2}\right)$$

which gives (using $\delta_{k+1} \leq \delta_k$ and ψ is decreasing) for every k

$$\int_{\delta_{k+1}}^{\delta_k} \frac{2\,dt}{t\,\psi(4/t^2)} \geq \frac{2}{\psi\left(4/\delta_k^2\right)} \int_{\delta_{k+1}}^{\delta_k} \frac{dt}{t} \geq 1$$

Summing over $k = 0, \dots, m-1$, we get.

$$\int_{\delta_m}^{\delta_0} \frac{2\,dt}{t\,\psi(4/t^2)} \geq m$$

By using the change of variable $u = 4/t^2$, we finally get

$$\int_4^{4/\delta_m^2} \frac{du}{u\,\psi(u)} \geq m = \int_4^{4/\varepsilon_m} \frac{du}{u\,\psi(u)}$$

Hence $\delta_m^2 \leq \varepsilon_m$ and Lemma 1 concludes that

$$p_{2m}(0,0) \leq \delta_m^2 \leq \varepsilon_m \qquad \blacksquare$$