# CHAPTER 4

# SPECTRAL RADIUS

$G = (V, E)$ transitive, loc. finite, connected, infinite graph
degree $d$, fixed origin $o \in V$

$(X_n)_{n \in \mathbb{N}}$ SRW starting from $x$, under $\mathbb{P}_x$.
$\left( p(x,y) = \frac{1}{d} \mathbb{1}_{x \sim y} \right)$

The goal of the chapter is to prove the following theorem.

**Thm [Kesten '58]**

The following are equivalent
(i) $\exists c > 0$ s.t. $\forall n \geq 0$ $\mathbb{P}_o[X_n = o] \leq e^{-cn}$

(ii) $G$ is non amenable.

In the previous chapter we already proved (ii) $\Rightarrow$ (i). In this chapter we will prove the other direction. We will rely on tools from operator theory.

RW $\longleftrightarrow$ operator $P$

exp. decay $\longleftrightarrow$ norm of the operator $< 1$.

## 1 SPECTRAL RADIUS

**Def.** (spectral radius of the SRW)

$$\boxed{\rho = \rho(G) := \limsup_{n\to\infty} p_n(o)^{\frac{1}{n}}}$$

(i.e. $\rho(G)$ is the smallest positive number s.t. $p_n(o) \leq \rho(G)^{n+o(n)}$)

**Rks** : • exponential decay of $p_n(o) \iff \rho(G) < 1$

• "limsup" is important if $G$ is bipartite ($p_{2n+1}(o) = 0$)

**Prop.** $\forall n \geq 0 \quad p_n(o) \leq \rho(G)^n$

**Pf:** Let $n \geq 0$. By Chapman Kolmogorov.

$\forall k \geq 0 \quad p_n(o)^k \leq p_{kn}(o)$

Therefore
$$p_n(o)^{\frac{1}{n}} \leq \limsup_{k\to\infty} p_{kn}(o)^{\frac{1}{kn}} \leq \rho(G) \quad \blacksquare$$

**Exercises:** Prove that :

- $\rho(G) = \lim_{n\to\infty} p_{2n}(o)^{\frac{1}{2n}}$,
- $\rho(G) = \lim_{n\to\infty} p_n(o)^{\frac{1}{n}}$ if $G$ not bipartite,
- $\frac{1}{d} \leq \rho(G) \leq 1$.

## 2 TRANSITION OPERATOR

"infinite" matrix $\longleftrightarrow$ associated linear operator.

$$P = \big(p(x,y)\big)_{x,y \in V}$$

$$P : \mathbb{R}^V \longrightarrow \mathbb{R}^V$$
$$f \longmapsto Pf$$

where $(Pf)(x) := \sum_{y \in V} p(x,y)\, f(y)$

<u>Not.</u> $\mathcal{C} := \{ f : V \to \mathbb{R} \text{ with finite support} \}$

For $f, g \in \mathcal{C}$ $\quad \langle f, g \rangle := \sum_{x \in V} f(x)\, g(x) \qquad \|f\| := \sqrt{\langle f, f \rangle}$

<u>Def.</u> The <u>transition operator</u> $P : \mathcal{C} \to \mathcal{C}$ associated to the SRW on $G$ is defined by

$$\forall f \in \mathcal{C} \quad \forall x \in V \qquad (Pf)(x) = \sum_{y \in V} p(x,y)\, f(y)$$

<u>Proba. interpretation</u>:

$$\boxed{Pf(x) = E_x[f(X_1)]}$$

Furthermore $\forall n \geq 0$ $\quad P^n f(x) = \sum_{y_1, \dots, y_n \in V} p(x,y_1) \cdots p(y_{n-1}, y_n)\, f(y_n)$ (by induction)

$$= \sum_{y \in V} \Big( \sum_{y_1, \dots, y_{n-1} \in V} p(x,y_1) \cdots p(y_{n-1}, y) \Big) f(y)$$

$$= \sum_{y \in V} p_n(x,y)\, f(y)$$

Therefore $\boxed{P^n f(x) = \mathbb{E}_x[f(X_n)]}$.

In particular, we have the following expression for the n-step transition probabilities:

$\forall n \geq 0\ \forall x, y \in V$ $\boxed{p_n(x,y) = \langle P^n \mathbb{1}_x, \mathbb{1}_y \rangle}$.

Rk: The operator $P$ can be naturally defined on the Hilbert space $\ell^2(V) = \{ f : V \to \mathbb{R} \quad \sum_{x \in V} f(x)^2 < \infty \}$

In this course we work with $\mathcal{E}$ to avoid convergence questions.

Prop. (i) $P$ is self-adjoint: $\forall f, g \in \mathcal{E}$ $\langle Pf, g \rangle = \langle f, Pg \rangle$

(ii) $\|P\| = \sup_{f \in \mathcal{E} \setminus \{0\}} \dfrac{\langle Pf, f \rangle}{\langle f, f \rangle}$ "Rayleigh Quotient"

Proof. (i)
$$\langle Pf, g \rangle = \sum_{x \in V} (Pf(x))\, g(x)$$
$$= \sum_{x,y \in V} p(x,y) f(y) g(x)$$
$$= \sum_{x,y \in V} p(y,x) f(y) g(x) \qquad \text{(Reversibility)}$$
$$= \sum_{y \in V} f(y) (Pg)(y)$$
$$= \langle f, Pg \rangle$$

(ii)

Let $C = \sup_{f \in \ell_k \setminus \{0\}} \frac{|\langle Pf, f \rangle|}{\langle f, f \rangle}$

By Cauchy Schwarz, $\forall f \neq 0$ $|\langle Pf, f \rangle| \leq \|P\|_2 \|f\|_2^2$

Hence $C \leq \|P\|_2$

Now set $g = \frac{\|f\|}{\|Pf\|} \cdot Pf$

$$\|Pf\| \|f\| = \langle Pf, g \rangle$$

$$= \frac{1}{4}\left(\langle P(f+g), f+g \rangle - \langle P(f-g), f-g \rangle\right)$$

$$\leq C \cdot \frac{1}{4}\left(\|f+g\|^2 + \|f-g\|^2\right)$$

$$\leq C \|f\|^2$$

Hence $\|P\|_2 \leq C$

Finally observe that $|\langle Pf, f \rangle| = \left|\sum_x (Pf)(x) \, f(x)\right|$

$$= \left|\sum_x E_x[f(X_1)] \, f(x)\right|$$

$$\leq \sum_x E_x[|f(X_1)|] \, |f(x)|$$

$$= \langle P|f|, |f| \rangle$$

And therefore $C = \sup_{f \neq 0} \frac{\langle Pf, f \rangle}{\|f\|^2}$ ■

## 3 SPECTRAL RADIUS AND OPERATOR NORM

<u>Th</u> $\rho(G) = \|P\|$

<u>Proof:</u> Consider the finite-dimensional space (for $k \geq 0$)

$\mathcal{C}_k = \{ f \in \mathcal{C} : \operatorname{Supp} f \subset B_k \}$, equipped with $\langle f, g \rangle_{\mathcal{C}_k} = \langle f, g \rangle$

Define $P_k : \mathcal{C}_k \to \mathcal{C}_k$ by setting

$$\forall f \in \mathcal{C}_k \quad (P_k f)(x) = Pf(x) \, 1\!\!1_{x \in B_k}$$

$P_k$ is a self-adjoint linear operator on the Euclidean space $(\mathcal{C}_k, \langle \cdot, \cdot \rangle_{\mathcal{C}_k})$. Indeed

$$\forall f, g \in \mathcal{C}_k \quad \langle P_k f, g \rangle_{\mathcal{C}_k} \underset{g \in \mathcal{C}_k}{=} \langle P f, g \rangle \underset{P \text{ self. adj}}{=} \langle f, Pg \rangle \underset{f \in \mathcal{C}_k}{=} \langle f, P_k g \rangle_{\mathcal{C}_k}$$

Let $\lambda = \|P_k\|$ be the largest eigen-value of $P_k$. Let $\varphi$ be an eigenvector associated to $\lambda$.

$$\forall n \quad \lambda^n = |\langle P_k^n \varphi, \varphi \rangle| \leq \Big| \sum_{x, y \in B_k} (P_k^n)_{x,y} \, \varphi_x \varphi_y \Big|$$

$$\leq \sum_{x, y \in B_k} \underbrace{p_n(x,y)}_{\leq \rho(G)^n} |\varphi_x| |\varphi_y|$$

$$\leq |B_k| \, \|\varphi\|_\infty^2 \times \rho(G)^n$$

Therefore $\|P_k\| \leq \rho(G)$ and the inequality $\|P\| \leq \rho(G)$ follows from $\|P_k\| \xrightarrow[k \to \infty]{} \|P\|$.

The other inequality follows from Cauchy - Schwarz:

$$p_n(o)^{\frac{1}{n}} = \langle P^n 1\!\!1_o , 1\!\!1_o \rangle^{\frac{1}{n}} \leq \|P\| ,$$

which implies $\rho(G) \leq \|P\|$ ∎

## 4 PROOF OF KESTEN THEOREM

Reminder $\Phi$ = isop. constant of $G$

Thm:

$$1 - \frac{\Phi}{d} \leq \rho(G) \leq \sqrt{1 - \frac{1}{16}\left(\frac{\Phi}{d}\right)^2}$$

Proof. (Lower bound)

Let $K \subset V$ finite, $f := 1\!\!1_K$

$$\langle Pf , f \rangle = \sum_{x \in K} \mathbb{P}_x [X_1 \in K]$$

$$= |K| - \sum_{x \in K} \mathbb{P}_x [X_1 \notin K]$$

$$= |K| - \frac{1}{d} |\partial K|$$

$$\rho(G) = \|P\| \geq \frac{\langle Pf , f \rangle}{\langle f , f \rangle} = 1 - \frac{1}{d} \frac{|\partial K|}{|K|} .$$

Therefore by taking the supremum of the RHS over $K$ finite we get $\rho(G) \geq 1 - \frac{\Phi}{d}$ ∎

(upper bound)

$p_{2n}(0) \leq 2\, q_{2n}(0)$ where $q_n$ is the $n$-step transition prob. for the LRW.

Now by Chapter 3,

$$q_{2n}(0) \leq \left(1 - \frac{\Phi^2}{16d^2}\right)^n$$

Therefore
$$\underbrace{p_{2n}(0)^{\frac{1}{2n}}}_{\downarrow\, n\to\infty \atop \rho(G)} \leq \sqrt{2}\left(\sqrt{1-\frac{\phi^2}{16d^2}}\right)$$

$\blacksquare$