CHAPTER 4 :

THE VAROPOULOS - CARNE BOUND.

Ref: [LYONS-PERES] [Blog T. TAO]

$G = (V,E)$ transitive, locally-finite, connected, infinite graph
degree d, fixed origin $0 \in V$.

$(X_n)_{n\geq 0}$ SRW on G, P transition operator, $p_n(x,y) = \mathbb{P}_x[X_n = y]$.

Motivation: In the previous chapters, we have seem

$$p_n(x,y) \leq p_n(0,0) \leq \|P\|_2^n$$

We expect $p_n(x,y)$ to be be "substancially" smaller than $\|P\|_2^n$ when x and y are far from each other.
(In particular, we trivially have $p_n(x,y) = 0$ if $d(x,y) > n$)

In this chapter, we prove the following inequality, which improves quantitatively the bound $p_n(x,y) \leq \|P\|_2^n$ when $d(x,y) \geq \sqrt{2n}$.
It is a generalization and improvement by Carne (1985) of a result of Varopoulos (1985). The statement below is a refinement of Lyons and Peres (see Thm 13.4 in [LYONS-PERES]).

Thm: For all $x,y \in V$, for all $n \geq 0$, we have:

$$\boxed{p_n(x,y) \leq 2\,\|P\|^n \exp\left(-\frac{d(x,y)^2}{2n}\right)}$$

[Varopoulos - Carne Bound]

We will see several applications
- When G has polynomial growth it implies that $P_0[d(0,X_n) \leq C\sqrt{n\log n}] \to 1$
- When G has sub-exponential growth, it implies that X_n has speed $\ell = 0$.

1. The case $G = \mathbb{Z}$.

Thm: Let $(X_n)_{n \geq 0}$ be a SRW on $\mathbb{Z}$, starting at 0. Then

$$\boxed{\mathbb{P}_0 [\, |X_n| \geq d \,] \leq 2\, e^{-d^2/2n}.}$$

Proof: Let $Z_1, Z_2, \dots, Z_n$ be iid random variables s.t. $\mathbb{P}[Z_1 = 1] = \mathbb{P}[Z_1 = -1] = \frac{1}{2}$.

Let $s = \frac{d}{n}$. By symmetry

$$\mathbb{P}_0 [\, |X_n| \geq d \,] = 2\,\mathbb{P}\left[s Z_1 + \dots + s Z_n \geq s^2 n \right]$$

$$= 2\,\mathbb{P}\left[e^{s Z_1 + \dots + s Z_n} \geq e^{s^2 n} \right]$$

$$\leq 2 \left(\frac{E[e^{s Z_1}]}{e^{s^2}} \right)^n \qquad [\text{By Markov inequality + independence}]$$

Using that $E[e^{s Z_1}] = \cosh(s) = \sum_{k=0}^{\infty} \frac{1}{(2k)!} s^{2k} \leq \sum_{k=0}^{\infty} \frac{1}{2^k \times k!} s^{2k} = e^{s^2/2}$,

we finally get $\mathbb{P}_0 [\, |X_n| \geq d \,] \leq 2\, e^{-n s^2/2}$ ∎

2. Chebyshev polynomials.

Definition [Chebyshev Polynomials]

We consider the sequence of polynomials $(T_n)_{n \in \mathbb{Z}}$ defined by

$$\begin{cases} T_0(x) = 1 \\ T_1(x) = x \\ T_{n+2}(x) = 2x\, T_{n+1}(x) - T_n(x) \end{cases} \qquad \text{and } T_{-n} = T_n$$

Property: $\deg(T_k) = |k|$ and For every $\theta \in \mathbb{R}$ $\boxed{\cos(n\theta) = T_n(\cos\theta)}$

Proof: By induction (use $\cos((n+1)\theta) + \cos((n-1)\theta) = 2\cos\theta \cos n\theta$) ∎

Prop: Let $q_n(k) = \mathbb{P}_0[X_n = k]$ for the SRW on $\mathbb{Z}$. $(n \in \mathbb{N}, k \in \mathbb{Z})$

We have, for every n

$$X^n = \sum_{k=-n}^{n} q_n(k) \cdot T_k(X)$$

Proof. It suffices to prove the equality for $X = \cos\theta$, $\theta \in \mathbb{R}$.

$$(\cos\theta)^n = \left(\frac{e^{i\theta} + e^{-i\theta}}{2}\right)^n = \sum_{\ell=0}^{n} \underbrace{\frac{1}{2^n}\binom{n}{\ell}}_{= q_n(2\ell - n)} e^{i\theta(2\ell - n)}$$

$$= \sum_{k=-n}^{n} q_n(k)\, e^{i\theta k}$$

sym $q_n(k) = q_n(-k)$

$$= \sum_{k=-n}^{n} q_n(k)\, \frac{e^{i\theta k} + e^{-i\theta k}}{2}$$

$$= \sum_{k=-n}^{n} q_n(k)\, T_k(\cos\theta)$$

Prop. For $Q \in \mathbb{R}[X]$, write $\|Q\|_\infty = \max_{x \in [-1,1]} |Q(x)|$

$$\min_{Q \in \mathbb{R}_{k-1}} \|X^n - Q\|_\infty \leq 2\, e^{-k/2n^2}$$

Proof. Let $Q = \sum_{\ell \leq k-1} q_n(\ell)\, T_\ell$

For every $\theta \in \mathbb{R}$, we have

$$\left|(\cos\theta)^n - Q(\cos\theta)\right| \underset{\text{prop}}{=} \left| \sum_{\ell : |\ell| \geq k} q_n(\ell) \cos(\theta . \ell) \right|$$

$$\leq \sum_{\ell : |\ell| \geq k} q_n(\ell)$$

$$= \mathbb{P}_0^{\mathbb{Z}} \left[|X_n| \geq k \right] \leq 2 e^{-k^2/n} \quad \blacksquare$$

3 POLYNOMIALS AND OPERATORS

Prop. Let $S : \mathcal{E} \to \mathcal{E}$ self adjoint bounded ($\|S\| < \infty$)
Let $Q \in \mathbb{R}[X]$. We have

$$\|Q(S)\| \leq \max_{|u| \leq \|S\|} |Q(u)|$$

Proof. 1st proof. By the spectral theorem for bounded s.a. operators [Rudin, Funct. analysis p. 308 / 303]

Writing $d\mu$ the spectral measure of S attached to f,

$$|\langle Q(S) f, Q(S) f \rangle| = \left| \int_{-\|S\|}^{\|S\|} Q(u)^2 \, d\mu(u) \right|$$

$$\leq \max_{|u| \leq \|S\|} Q(u)^2 \times \underbrace{\int_{-\|S\|}^{\|S\|} d\mu(u)}_{=1}$$

Second proof (with a reduction to finite dimension)

Consider the finite-dimensional space $\mathcal{C}_n = \{ f \in \mathcal{C} : \mathrm{Supp}(f) \subset B_n \}$ equiped with the inner product $\langle f, g \rangle_{\mathcal{C}_n} = \langle f, g \rangle$

Define $S_n : \mathcal{C}_n \to \mathcal{C}_n$ by setting

$$\forall f \in \mathcal{C}_n \qquad (S_n f)(x) = (S f(x)) \mathbb{1}_{x \in B_n}$$

S_n is a self-adjoint linear operator on the (finite dimensional) Euclidean space $(\mathcal{C}_n, \langle \cdot, \cdot \rangle_{\mathcal{C}_n})$. Indeed

$$\forall f, g \in \mathcal{C}_n \quad \langle S_n f, g \rangle_{\mathcal{C}_n} \underset{\substack{\uparrow \\ g \in \mathcal{C}_n}}{=} \langle S f, g \rangle \underset{\substack{\uparrow \\ S \text{ auto-adjoint}}}{=} \langle f, S g \rangle \underset{\substack{\uparrow \\ f \in \mathcal{C}_n}}{=} \langle f, S_n g \rangle_{\mathcal{C}_n}.$$

Therefore there exists an orthonormal basis $(\varphi_1, \dots, \varphi_L)$ of $\mathcal{C}_n$ made of eigenvectors of S_n. Writing λ_ℓ for the eigenvalue associated to φ_ℓ, we obtain for every $f \in \mathcal{C}_k$ and n large,

$$Q(S_n) f = Q(S_n) \Big(\sum_{\ell \leq L} \langle f, \varphi_\ell \rangle \varphi_\ell \Big)$$

$$= \sum_{\ell \leq L} \langle f, \varphi_\ell \rangle Q(\lambda_\ell) \varphi_\ell$$

Hence $\| Q(S_n) f \|^2 = \sum_\ell \langle f, \varphi_\ell \rangle^2 Q(\lambda_\ell)^2 \leq \Big(\max_{|\lambda| \leq \|S_n\|} |Q(\lambda)| \Big)^2 \|f\|^2$

Which implies $\| Q(S_n) \| \leq \max_{|\lambda| \leq \|S_n\|} |Q(\lambda)|$.

and the proof follows from the fact that $\|Q(S_n) f\|_2 \xrightarrow[n \to \infty]{} \|Q(S) f\|_2$ and $\|S_n\|_2 \leq \|S\|_2$.

4 PROOF OF VAROPOULOS-CARNE BOUND

Let $x \in V$.

Key observation. Set $k := |x|$ $\quad (= d(o,x))$

$$\forall \ell < k \quad \underbrace{\langle P^\ell \mathbb{1}_o , \mathbb{1}_x \rangle}_{p_\ell(x)} = 0$$

By linearity, this implies

$$\forall Q \in \mathbb{R}_{k-1}[X] \quad \langle Q(P)\mathbb{1}_o, \mathbb{1}_x \rangle = 0$$

Set $\bar{P} = \frac{1}{\|P\|} \cdot P$. For every $Q \in \mathbb{R}_{k-1}(X)$, we have

$$\begin{aligned}
\langle P^n \mathbb{1}_o, \mathbb{1}_x \rangle &= \|P\|^n \langle \bar{P}^n \mathbb{1}_o, \mathbb{1}_x \rangle \\
&= \|P\|^n \langle (\bar{P}^n - Q(\bar{P}))\mathbb{1}_o, \mathbb{1}_x \rangle \\
&\leq \|P\|^n \, \|\bar{P}^n - Q(\bar{P})\| && [\text{Cauchy-Schwarz}] \\
&\leq \|P\|^n \cdot \max_{|u| \leq 1} |u^n - Q(u)| && [\text{Prop.}]
\end{aligned}$$

Hence
$$\begin{aligned}
\langle P^n \mathbb{1}_o, \mathbb{1}_x \rangle &\leq \|P\|^n \times \min_{Q \in \mathbb{R}_{k-1}[X]} \|X^n - Q(X)\|_\infty \\
&\leq \|P\|^n \times 2e^{-k^2/2n} && [\text{Prop.}]
\end{aligned}$$

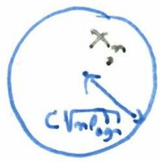
4 ONE APPLICATION.

The Varopoulos-Carne bound provides useful upper bounds on the speed at which the random walk moves away from 0. Write $|X_n| = d(0, X_n)$. When the graph G has sub-exponential volume growth, we obtain substancial improvements to the trivial bound $|X_n| \leq n$. In particular we have the following bounds when the growth is polynomial or stretch exponential:

Corollary (to Varopoulos-Carne bound)

(i) If G has polynomial volume growth ($\exists A$ $|B_n| \leq A n^D$) then $\exists C < \infty$ s.t.

$$\limsup_{n \to \infty} \frac{|X_n|}{\sqrt{n \log n}} \leq C \qquad \mathbb{P}_0\text{-a.s.}$$



(ii) If $|B_n| \leq A e^{C n^\alpha}$ for some $A, C > 0$ and $0 < \alpha \leq 1$, then

$$\limsup \frac{|X_n|}{n^{1/(2-\alpha)}} \leq (2C)^{1/(2-\alpha)} \qquad \mathbb{P}_0\text{-a.s.}$$

Proof: (i) Assume $|B_n| \leq A n^D$. Let $N = C\sqrt{n \log n}$

$$\mathbb{P}[|X_n| \geq N] \leq \sum_{k \geq N} \mathbb{P}_0[|X_n| = k]$$

$$\leq \sum_{k \geq N} |B_k| \cdot 2 \exp\left(-\frac{k^2}{2n}\right)$$

$$\leq 2A \underbrace{\sum_{k \geq N} k^D \exp\left(-\frac{k^2}{2n}\right)}_{\leq C_1 N^{D+1} \exp\left(-\frac{N^2}{2n}\right)}$$

Hence $P\left[|X_n| \geq C\sqrt{n \log n} \right] \leq 2AC_1 \times (n \log n)^{\frac{D+1}{2}} n^{-\frac{C^2}{2}}$

If $C > \sqrt{D+3}$ $\quad \sum_n \mathbb{P}_0 \left[\frac{|X_n|}{\sqrt{n \log n}} \geq C \right] < \infty$

And Borel-Cantelli lemma concludes that

$$\limsup_{n\to\infty} \frac{|X_n|}{\sqrt{n \log n}} \leq C \quad \text{a.s.}$$

(ii) exercice.

Remarks:

- In (i), it is possible to choose $C = \sqrt{D+1}$ (exercice)
- The bound (i) is not sharp for $G = \mathbb{Z}^d$. Indeed by the law of the iterated logarithm

$$\limsup_{n\to\infty} \frac{|X_n|}{\sqrt{n \log \log n}} < \infty.$$