Examples

(2) Let $f(x) = \cos(x)$ for $x \in \mathbb{R}$. Then we have seen that $f'(x) = -\sin(x)$. We also know from the construction of the number π that $\sin(x) > 0$ on the interval $]0, \pi[$. So the function cosine restricted to $[0, \pi]$ is strictly decreasing (Kor. 4.2.5). It follows that cosine gives by restriction a bijection

$$[0, \pi] \longrightarrow [-1, 1]$$

(since $\cos(0) = 1$, $\cos(\pi) = -1$). The reciprocal bijection is denoted arccos:

$$\arccos : [-1, 1] \longrightarrow [0, \pi]$$

The function arccos is a continuous, strictly decreasing bijection. Since $\cos'(x) \neq 0$ for $0 < x < \pi$, the arccos is also differentiable on $]-1, 1[$, and

$$\arccos'(x) = \frac{1}{\cos'(\arccos(x))} \qquad \text{(Satz 4.1.12)}$$

$$= -\frac{1}{\sin(\arccos(x))}.$$

However note that this can be simplified: since we know that $\cos(t)^2 + \sin(t)^2 = 1$ for all t, we have $\sin(t) = \sqrt{1-\cos(t)^2}$ or $-\sqrt{1-\cos(t)^2}$.

If $t = \arccos(x)$, then $0 \leq t \leq \pi$, and so $\sin(\arccos(x))$ is ≥ 0 $[\sin(t) \geq 0$ if $0 \leq t \leq \pi]$, which means that

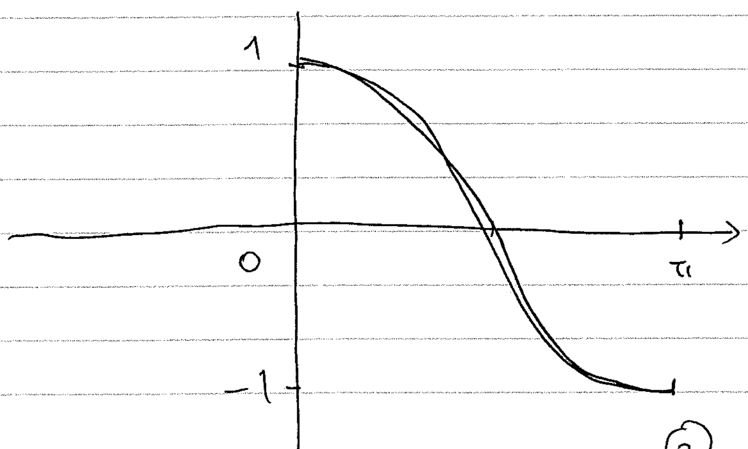
$$\sin(\arccos(x)) = \sqrt{1-\cos^2(\arccos(x))}$$
$$= \sqrt{1-x^2}$$

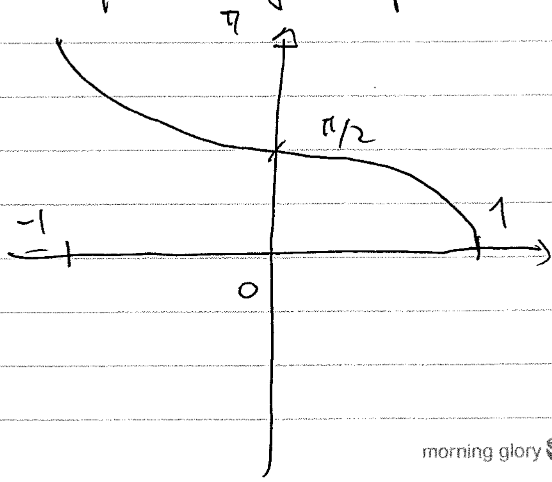
(because cos / arccos are reciprocal bijections)

Hence $$\arccos'(x) = -\frac{1}{\sqrt{1-x^2}}.$$

Note: it is true that $\cos(\arccos(x)) = x$ for all $x \in [-1, 1]$, but $\arccos(\cos(x)) = x$ only if $0 \leq x \leq \pi$! For instance $\arccos(\cos(2\pi)) = \arccos(1) = 0$.

The graphs have more or less the following shapes:



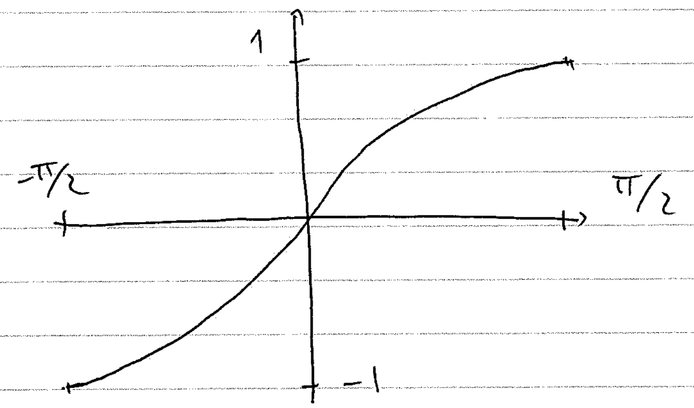


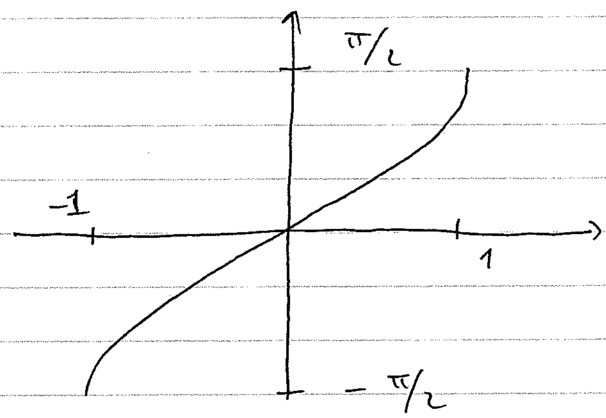
(3) Similarly, one proves that the function sine is a bijection $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right] \longrightarrow [-1, 1]$, strictly increasing, with reciprocal bijection

$$\arcsin : [-1, 1] \longrightarrow \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$$

also strictly increasing, and

$$\arcsin'(x) = \frac{1}{\sqrt{1-x^2}}.$$





(4) The function $f(x) = \frac{\sin(x)}{\cos(x)}$, defined for all x such that $\cos(x) \neq 0$ [which are $x = \frac{\pi}{2}, -\frac{\pi}{2}, \frac{3\pi}{2}, \frac{5\pi}{2}, \ldots, -\frac{3\pi}{2}$, is called the <u>tangent</u> function. We consider it on the interval $\left]-\frac{\pi}{2}, \frac{\pi}{2}\right[$. On this interval, the function is continuous and differentiable, as quotient of differentiable functions (with non-zero denominator).

(Satz 4.1.9)

We have

$$\tan'(x) = \frac{\sin'(x)\cos(x) - \cos'(x)\sin(x)}{\cos(x)^2}$$

$$= \frac{\cos(x)^2 + \sin(x)^2}{\cos(x)^2} = \frac{1}{\cos(x)^2}$$

(which is also equal to

$$\frac{\cos^2(x) + \sin^2(x)}{\cos^2(x)} = 1 + \tan(x)^2 \Big)$$

Since $\tan'(x) > 0$ for all x, the function tan is strictly ~~[illegible]~~ increasing for $-\frac{\pi}{2} < x < \frac{\pi}{2}$. We have

$$\lim_{\substack{x \to \frac{\pi}{2} \\ x < \frac{\pi}{2}}} \tan(x) = \lim_{\substack{x \to \frac{\pi}{2} \\ x < \frac{\pi}{2}}} \frac{\overbrace{\sin(x)}^{\to 1}}{\underbrace{\cos(x)}_{\to 0}} = +\infty$$

and is > 0 for $0 < x < \frac{\pi}{2}$

and $\lim_{\substack{x \to -\frac{\pi}{2} \\ x > -\frac{\pi}{2}}} \tan(x) = -\infty$. So we get a strictly increasing bijection

$$\left] -\frac{\pi}{2}, \frac{\pi}{2} \right[\longrightarrow \mathbb{R}.$$

The reciprocal bijection is called arctan:

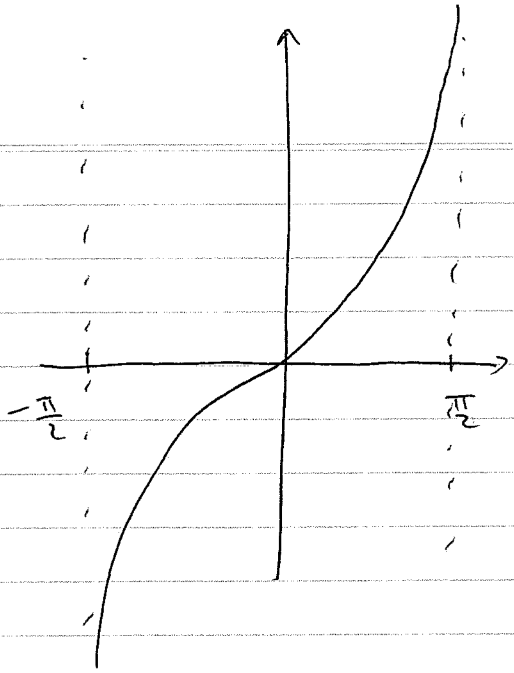
$$\arctan : \mathbb{R} \longrightarrow \left] -\frac{\pi}{2}, \frac{\pi}{2} \right[$$

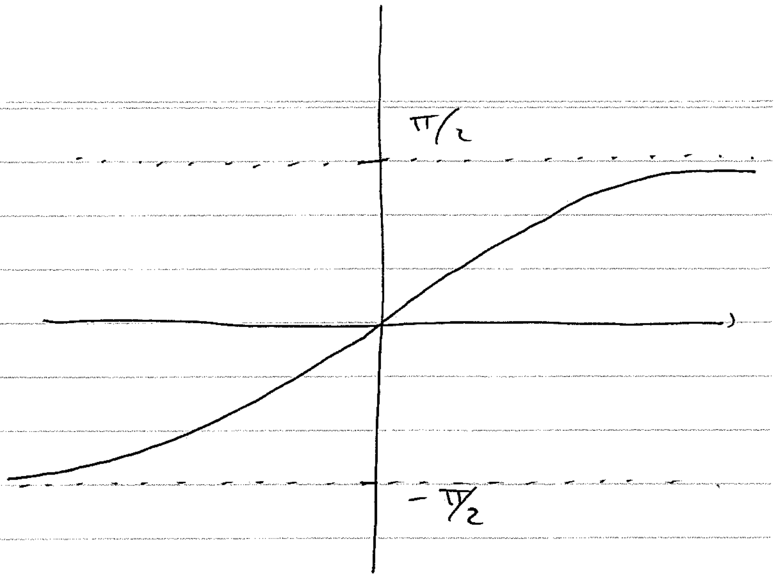
Since $\tan'(x) > 0$ for all x, arctan is differentiable with

$$\arctan'(x) = \frac{1}{\tan'(\arctan(x))}$$

$$= \frac{1}{1 + \tan(\arctan(x))^2}$$

$$= \frac{1}{1 + x^2}.$$



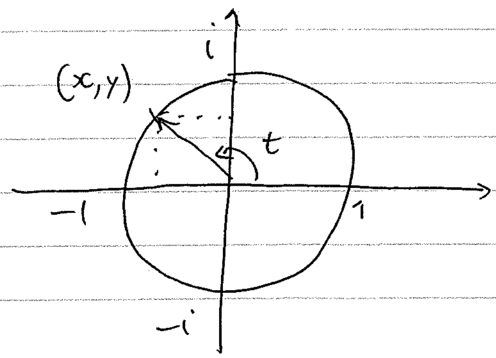


<u>Note:</u> $\tan(-x) = -\tan(x)$ and $\tan(x+\pi) = \tan(x)$.

(5) Finally note the example of $f(x) = x^3$ for $x \in \mathbb{R}$, which is <u>strictly</u> increasing on $\mathbb{R}$, <u>but</u> has a zero at $x = 0$. So $f'(x) > 0$ is <u>not</u> necessary to have a strictly increasing function.

<u>Application:</u> we can now deduce

<u>Th.</u> — Let $(x, y) \in \mathbb{R}^2$ be a point with $x^2 + y^2 = 1$ (Or $z \in \mathbb{C}$ with $|z| = 1$). Then there is a <u>unique</u> $t \in [0, 2\pi[$ such that $\begin{cases} x = \cos(t) \\ y = \sin(t) \end{cases}$ (or $z = e^{it}$)



<u>Proof.</u> Since $x^2 + y^2 = 1$, we get $0 \leq x^2 \leq 1$ so $-1 \leq x \leq 1$.

This means that there is a unique $u \in [0, \pi]$ such that $\cos(u) = x$ (Example (2)). From

$$1 = x^2 + y^2 = \cos^2(u) + \sin^2(u) = x^2 + \sin^2(u)$$

we get $y^2 = \sin^2(u)$, so either $y = \sin(u)$ or $y = -\sin(u)$.

<u>Case 1</u> - If $y \geq 0$ then $y = \sin(u)$ since $0 \leq u \leq \pi$. In that case, we can take $t = u$.

<u>Case 2</u> - If $y < 0$, then $y = -\sin(u) = \sin(2\pi - u)$.

But then also

trigonometric formulas

$$x = \cos(u) = \cos(2\pi - u)$$

so we can take $t = 2\pi - u \in [\pi, 2\pi]$.

The uniqueness (with $0 \leq t < 2\pi$) is left without proof (see the Skript for instance).

$\square$

<u>Remark</u> - The functions log, arcsin, arccos, arctan have the feature that their derivatives are "elementary" functions (e.g. $\log'(x) = \frac{1}{x}$), although they themselves are not "so" elementary. If we had a way to construct a function "from

its derivative", this would provide alternate definitions/construc-tions of these functions ... and this procedure is exactly what we will study in the next chapter.

4.3 Derivatives of higher order

Definition (4.3.1) – $I \subset \mathbb{R}$ interval

$f : I \longrightarrow \mathbb{R}$

$\cancel{n \geq 0}$

We define inductively: $f^{(0)} = f$, $f^{(1)} = f'$ if f is differentiable on I, $f^{(2)} = (f')' = f''$ if f' is differentiable on I, ..., $f^{(n+1)} = (f^{(n)})'$ if $f^{(n)}$ is differentiable on I.

We say that f is n-times continuously differentiable if $f, f', \dots, f^{(n)}$ exist, and if all of them are continuous.

We say that f is smooth if $f^{(n)}$ exists for all $n \geq 0$ (and is automatically continuous)

Note that $(f^{(n)})^{(m)} = f^{(n+m)}$, when defined.

Theorem (4.3.3) If $f, g : I \longrightarrow \mathbb{R}$ are n times differentiable, so are $f+g$ and fg, and

$$(f+g)^{(n)} = f^{(n)} + g^{(n)}$$

$$(fg)^{(n)} = \sum_{k=0}^{n} \binom{n}{k} f^{(k)} g^{(n-k)}$$ ("generalized Leibniz rule")

The proof is done by induction using the usual Leibniz rule $(fg)' = f'g + fg'$ and the formula

$$\binom{n}{k} = \binom{n-1}{k-1} + \binom{n-1}{k}$$

(a k element subset of $\{1, \dots, n\}$ is either a k element subset of $\{1, \dots, n-1\}$ or n together with a $k-1$ element subset of $\{1, \dots, n-1\}$).

Examples. (1) Polynomials and rational functions are smooth where they are defined.

(2) exp, sin, cos, are smooth on $\mathbb{R}$; log is smooth on $]0, +\infty[$.

(3) For instance: $\exp^{(n)} = \exp$ for all $n \geq 0$

$(\log)^{(n)} = \log$ if $n = 0$

$= \frac{1}{x}$, $n = 1$; $-\frac{1}{x^2}$, $n = 2$; $\frac{(-1)^{n-1}}{x^n}$, $n \geq 1$

Theorem (4.3.6) – If $f : I \longrightarrow J$ and $g : J \longrightarrow \mathbb{R}$ are n-times differentiable, then $g \circ f$ is also.

This is also easily checked by induction.

In particular, $f(x) = \frac{1}{x}$ is smooth on $]0, +\infty[$, so if $f : I \longrightarrow]0, +\infty[$ is smooth, so is $1/f$.

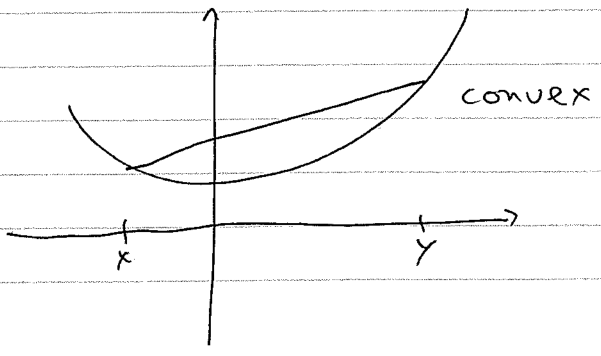
The second derivative has a special geometric meaning: it indicates whether a function is convex or concave or neither.

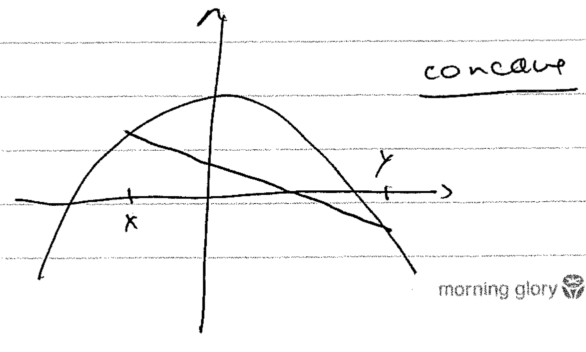
Definition – (4.2.13)

$I \subset \mathbb{R}$ interval, $f : I \longrightarrow \mathbb{R}$

f is convex on I if the graph of f on $[x, y]$ is always below the line joining $(x, f(x))$ and $(y, f(y))$, whenever $x \leq y$ are in I.

f is concave on I if the graph is above.

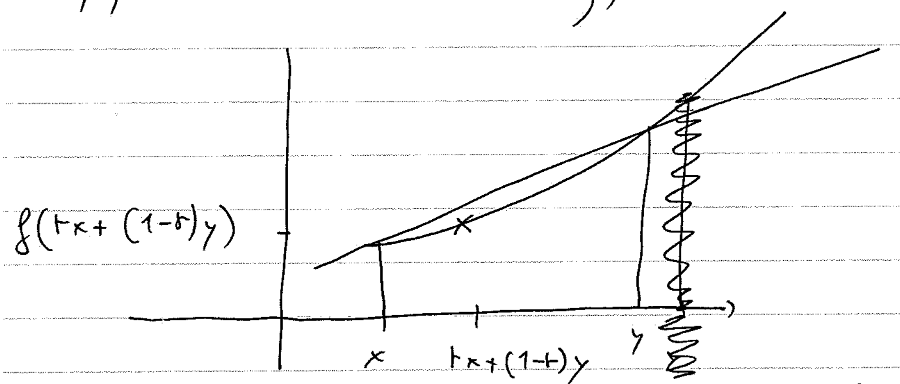




Analytically, ~~this~~ $f : I \longrightarrow \mathbb{R}$ is convex if and only if the inequality

$$f(tx + (1-t)y) \leq t f(x) + (1-t) f(y)$$

is true for $x \leq y$ in I and $0 \leq t \leq 1$. This is because the numbers $tx + (1-t)y$, $0 \leq t \leq 1$, range over all the interval $[x, y]$ exactly ($t = 0$ means y, $t = 1$ is x), and so $(tx + (1-t)y, \; t f(x) + (1-t) f(y))$



is the point over $tx + (1-t)y$ on the line joining $(x, f(x))$ and $(y, f(y))$.

By induction, one can then deduce that if $f : I \longrightarrow \mathbb{R}$ is convex, then

$$(*) \qquad f(t_1 x_1 + \dots + t_n x_n) \leq t_1 f(x_1) + \dots + t_n f(x_n)$$

for $n \geq 1$, $x_1, \dots, x_n$ in I, $t_1, \dots, t_n \geq 0$ and $t_1 + \dots + t_n = 1$.

Theorem (4.2.16) 2.17 f on $]a,b[$ two times differen-tiable. Then f is convex if and only if $f''(x) \geq 0$ for <u>all</u> x in $]a,b[$.

<u>Proof</u> See Skript.

<u>Example</u> (4.2.18) Let $f(x) = -\log(x)$ on $]0,+\infty[$.

Then $f'(x) = -\frac{1}{x}$, $f''(x) = \frac{1}{x^2} > 0$, so f is convex on $]0,+\infty[$.

This means in particular, taking $t_i = \frac{1}{n}$ for $1 \leq i \leq n$ in (*) that for $n \geq 1$ and $x_i > 0$, we get

$$-\log\left(\frac{x_1 + \cdots + x_n}{n}\right) \leq -\frac{1}{n}\log(x_1) \cdots - \frac{1}{n}\log(x_n)$$

so
$$\frac{1}{n}\log(x_1) + \cdots + \frac{1}{n}\log(x_n) \leq \log\left(\frac{x_1 + \cdots + x_n}{n}\right)$$

so
$$\underbrace{\exp\left(\frac{1}{n}\left(\log(x_1) + \cdots + \log(x_n)\right)\right)}_{=\sqrt[n]{x_1 \cdots x_n}} \leq \frac{x_1 + \cdots + x_n}{n}$$

$$\sqrt[n]{x_1 \cdots x_n} \leq \frac{x_1 + \cdots + x_n}{n}$$

("inequality of the geometric and arithmetic means").

morning glory