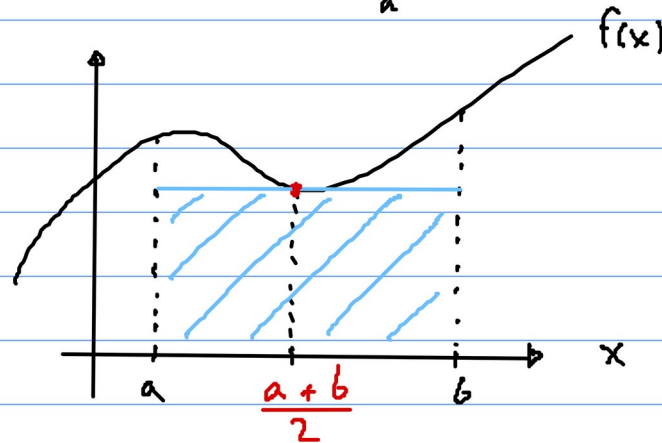


Bsp.: (3) Mittelpunktsregel (MR) ( $n=0$ )

$$\text{Knoten: } x_0 = \frac{a+b}{2}$$

$$\text{LP: } L_0^0(x) = 1$$

$$\text{Gewichte: } w_0 = \int_a^b L_0^0(x) dx = b-a$$



Damit

$$Q_0[f] = (b-a) \cdot f\left(\frac{a+b}{2}\right)$$

#### (4) Trapezregel (TR) ( $n=1$ )

Knoten:  $x_0 = a$ ,  $x_1 = b$

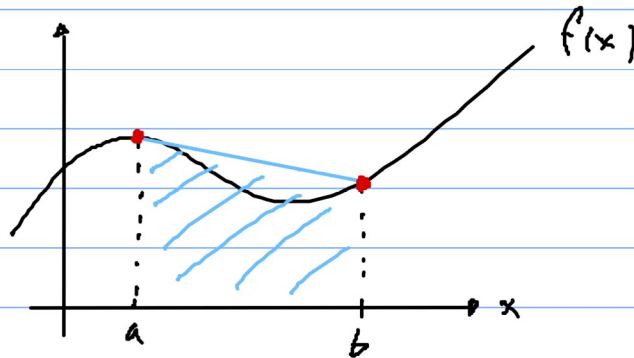
$$LP : L_0^1(x) = \frac{x-b}{a-b}$$

$$L_1^1(x) = \frac{x-a}{b-a}$$

$$\begin{aligned} \text{Gewichte: } w_0 &= \int_a^b L_0^1(x) dx = \int_a^b \frac{x-b}{a-b} dx \\ &= \frac{b-a}{2} \end{aligned}$$

$$L = \frac{(x-b)^2}{2(a-b)} \Big|_a^b =$$

$$w_1 = \dots = \frac{b-a}{2}$$



Damit

$$Q_1[f] = \frac{b-a}{2} (f(a) + f(b))$$

# (s) Simpson-Regel (SR) ( $n=2$ )

$$\text{Knoten: } x_0 = a, x_1 = \frac{a+b}{2}, x_2 = b$$

$$LP : L_0^2(x) = \frac{x - \frac{a+b}{2}}{a - \frac{a+b}{2}} \cdot \frac{x - b}{a - b}$$

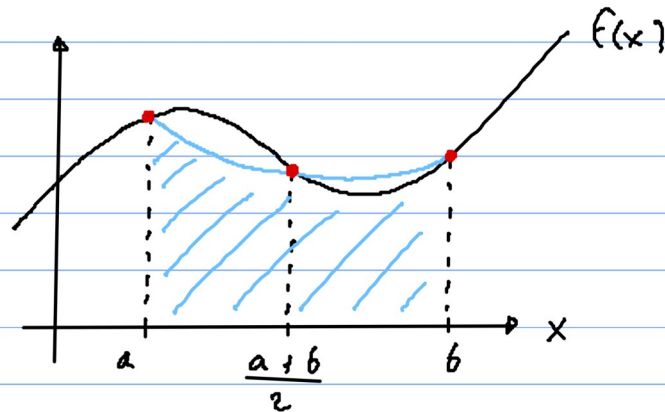
$$L_1^2(x) = \dots$$

$$L_2^2(x) = \dots$$

$$\text{Gewichte: } w_0 = \int_a^b L_0^2(x) dx = \dots = \frac{b-a}{6}$$

$$w_1 = \dots = \frac{4(b-a)}{6}$$

$$w_2 = \dots = \frac{b-a}{6}$$



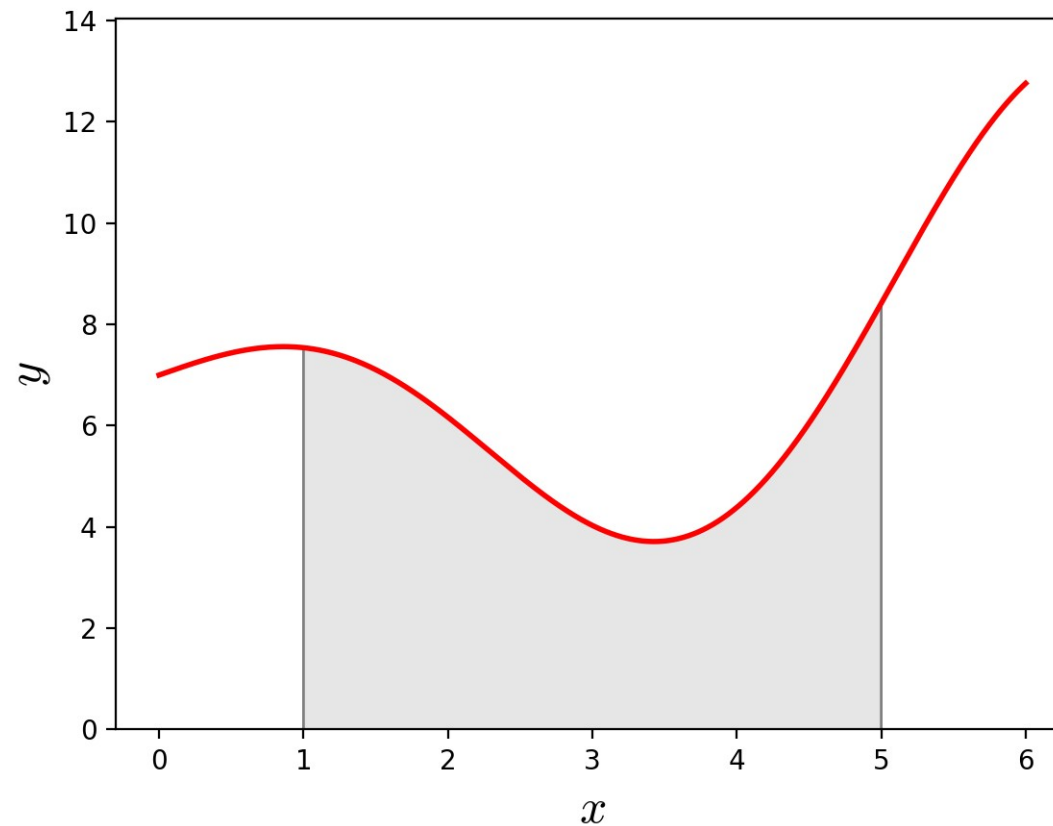
Damit

$$Q_2[f] = \frac{b-a}{6} \left( f(a) + 4 \cdot f\left(\frac{a+b}{2}\right) + f(b) \right)$$

Bsp.: (9)

# Quadratur-Regeln

$$f(x) = x \cos(x) + 7$$

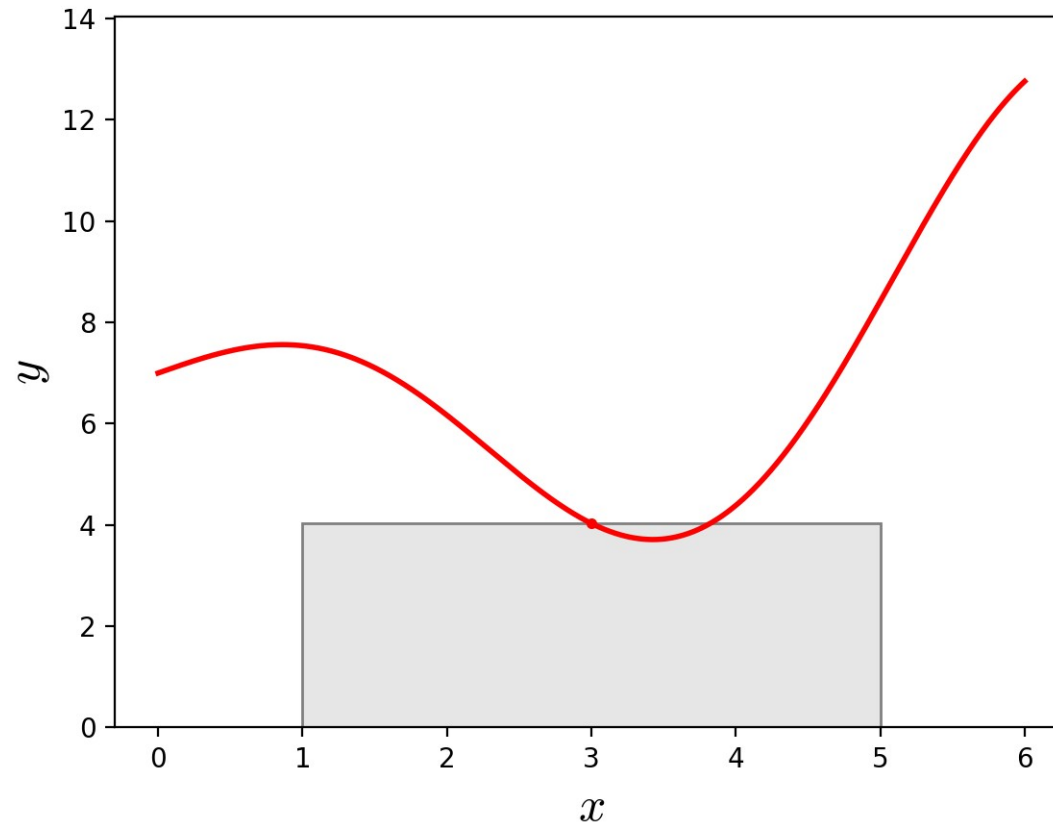


$$I[f] = \int_1^5 f(x) dx = 22.1073\dots$$

Bsp.: (9)

# Mittelpunktsregel

$$f(x) = x \cos(x) + 7$$

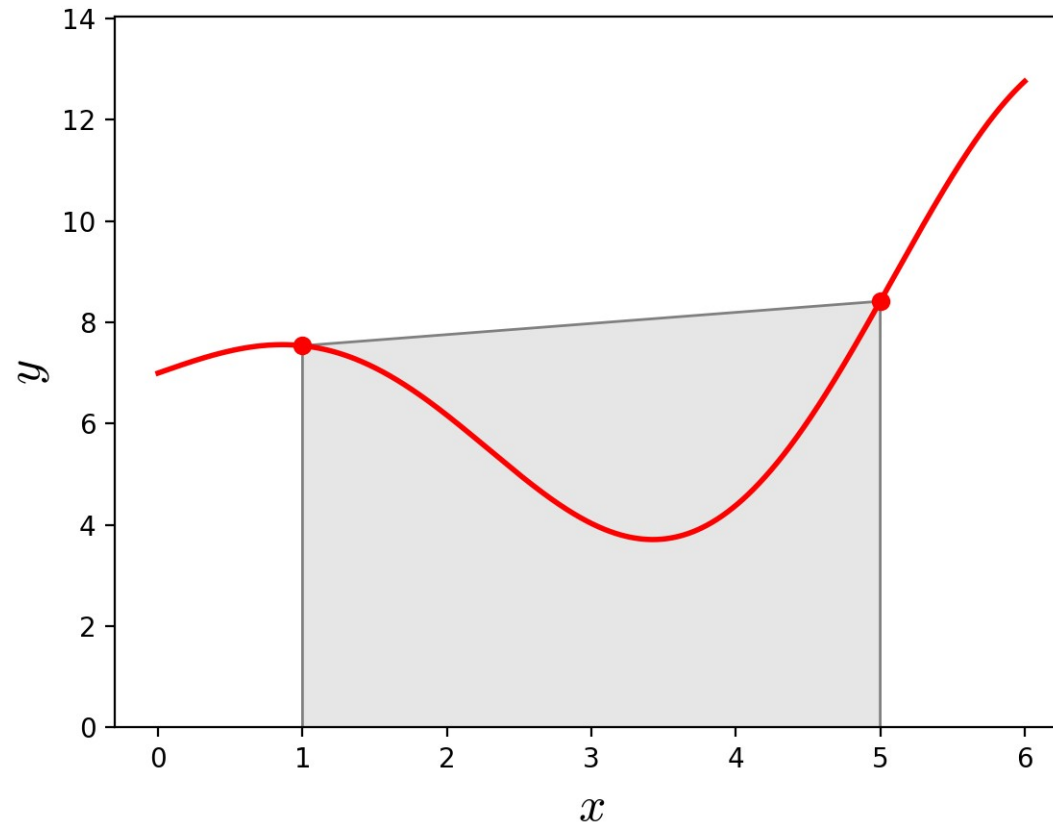


$$I[f] = 22.1073\dots \quad Q_0[f] = 16.1201\dots \quad E[f] = 5.9872\dots$$

Bsp.: (9)

# Trapezregel

$$f(x) = x \cos(x) + 7$$

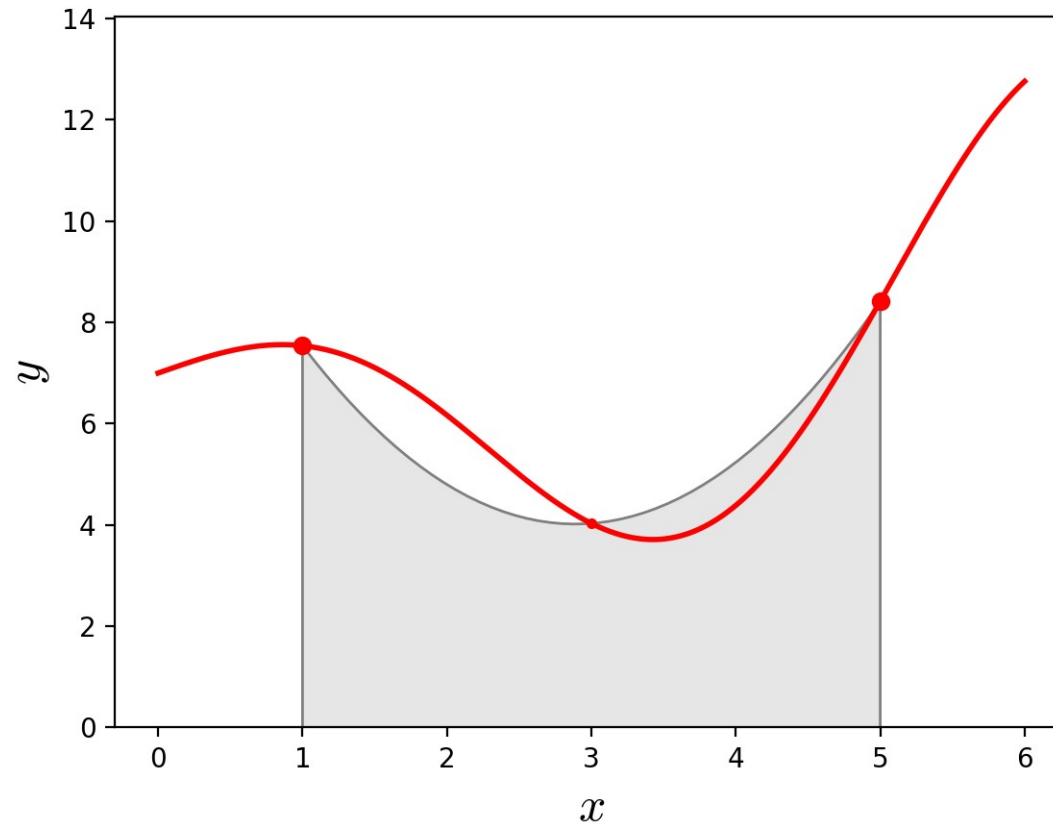


$$I[f] = 22.1073\dots \quad Q_1[f] = 31.9172\dots \quad E[f] = 9.8100\dots$$

Bsp.: (9)

# Simpson-Regel

$$f(x) = x \cos(x) + 7$$

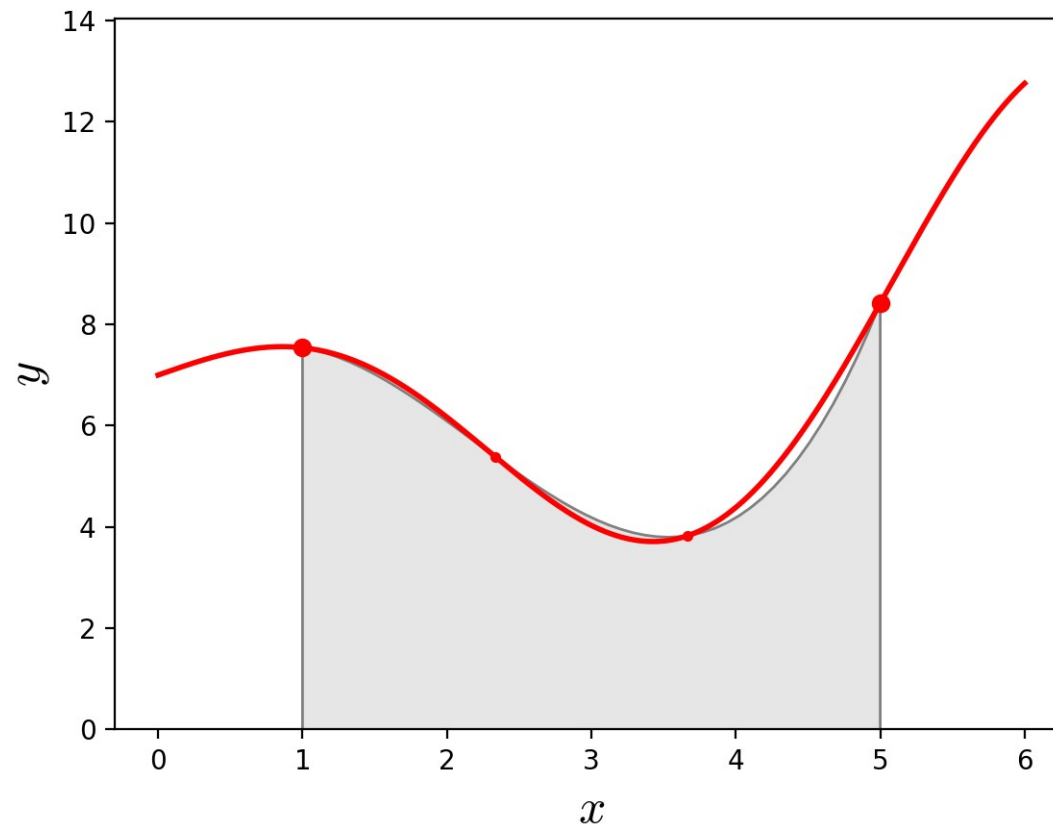


$$I[f] = 22.1073\dots \quad Q_2[f] = 21.3858\dots \quad E[f] = 0.7215\dots$$

Bsp.: (9)

# 3/8-Regel

$$f(x) = x \cos(x) + 7$$



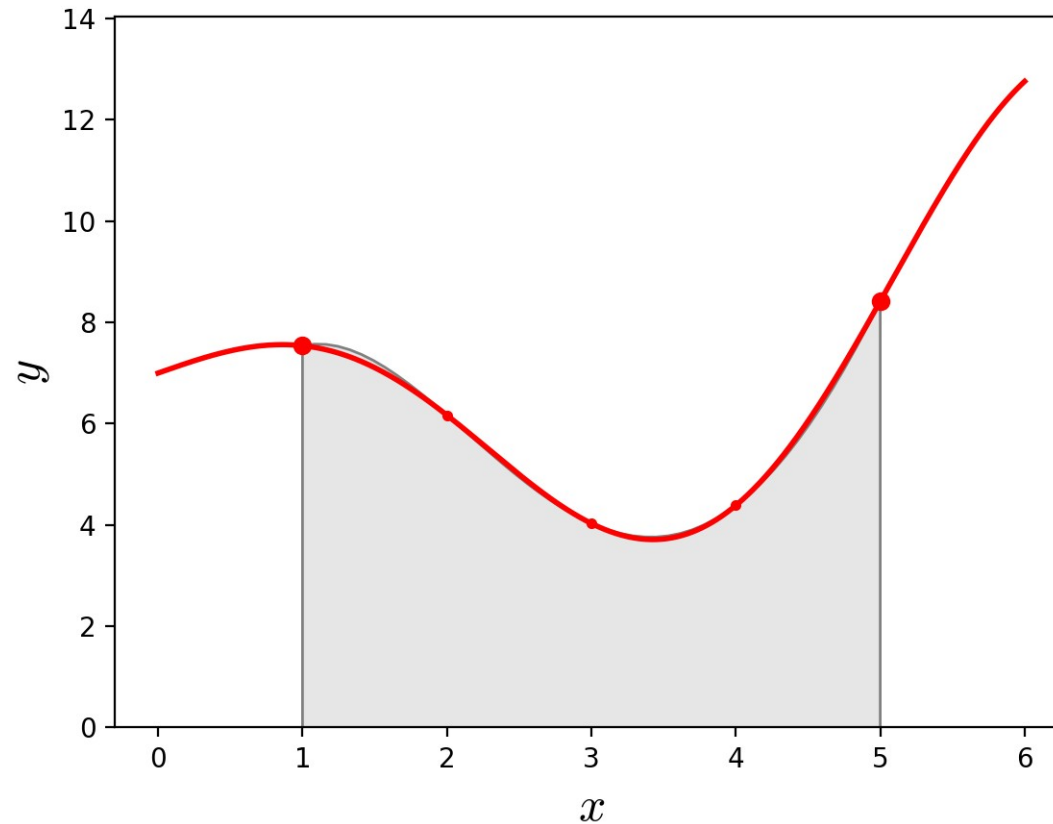
$$I[f] = 22.1073\dots \quad Q_3[f] = 21.8026\dots \quad E[f] = 0.3047\dots$$



Bsp.: (9)

# Milne-Regel

$$f(x) = x \cos(x) + 7$$

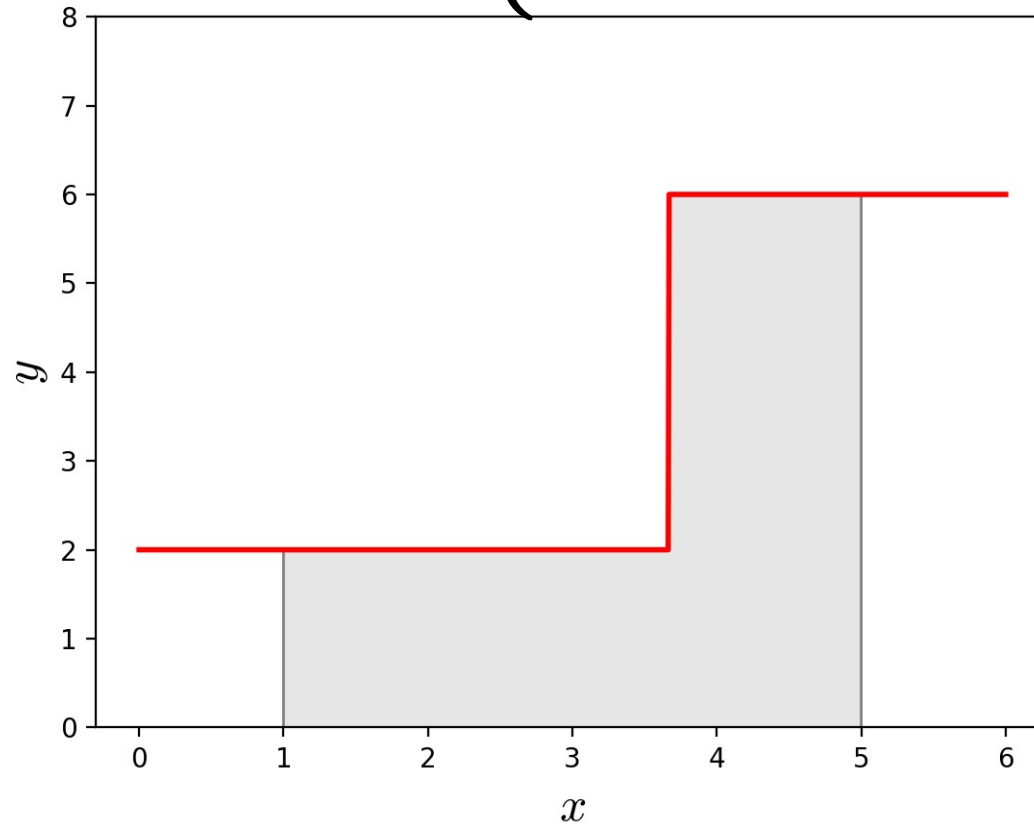


$$I[f] = 22.1073\dots \quad Q_4[f] = 22.1231\dots \quad E[f] = 0.0159\dots$$

Bsp.: (9)

# Quadrature

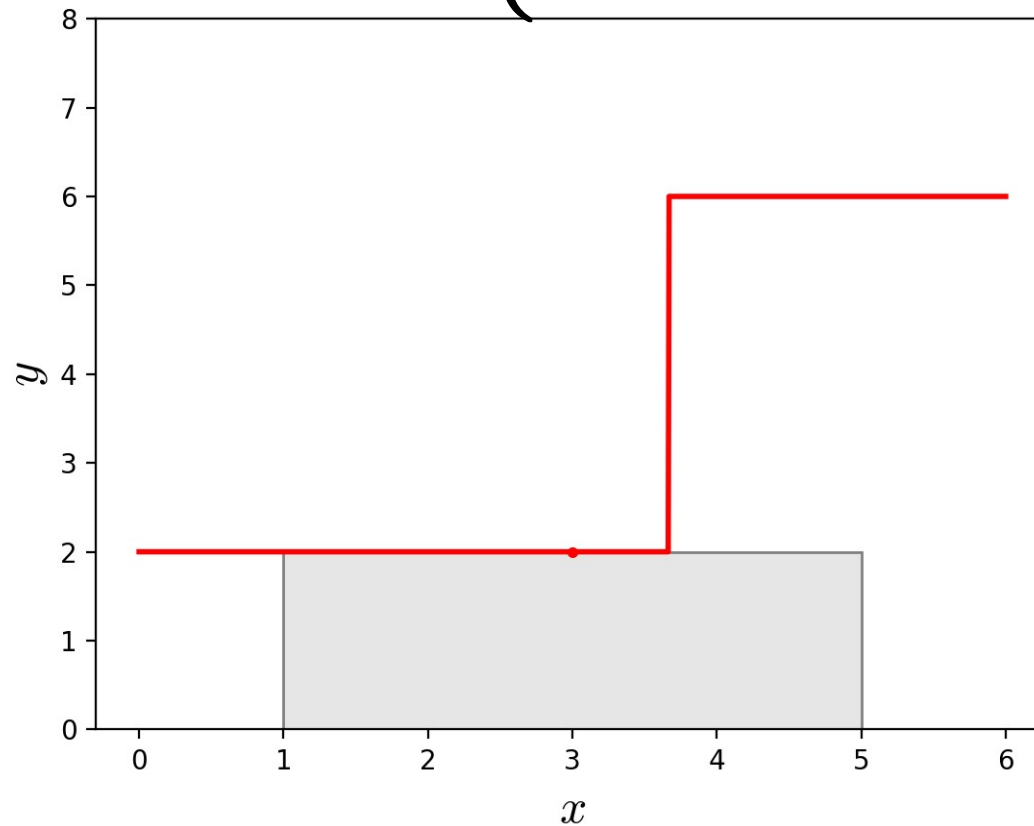
$$f(x) = \begin{cases} 2 & \text{if } x < \pi \\ 6 & \text{if } x \geq \pi \end{cases}$$



$$I[f] = \int_1^5 f(x) dx = 15.4336\dots$$

# Mittelpunktsregel

$$f(x) = \begin{cases} 2 & \text{if } x < \pi \\ 6 & \text{if } x \geq \pi \end{cases}$$

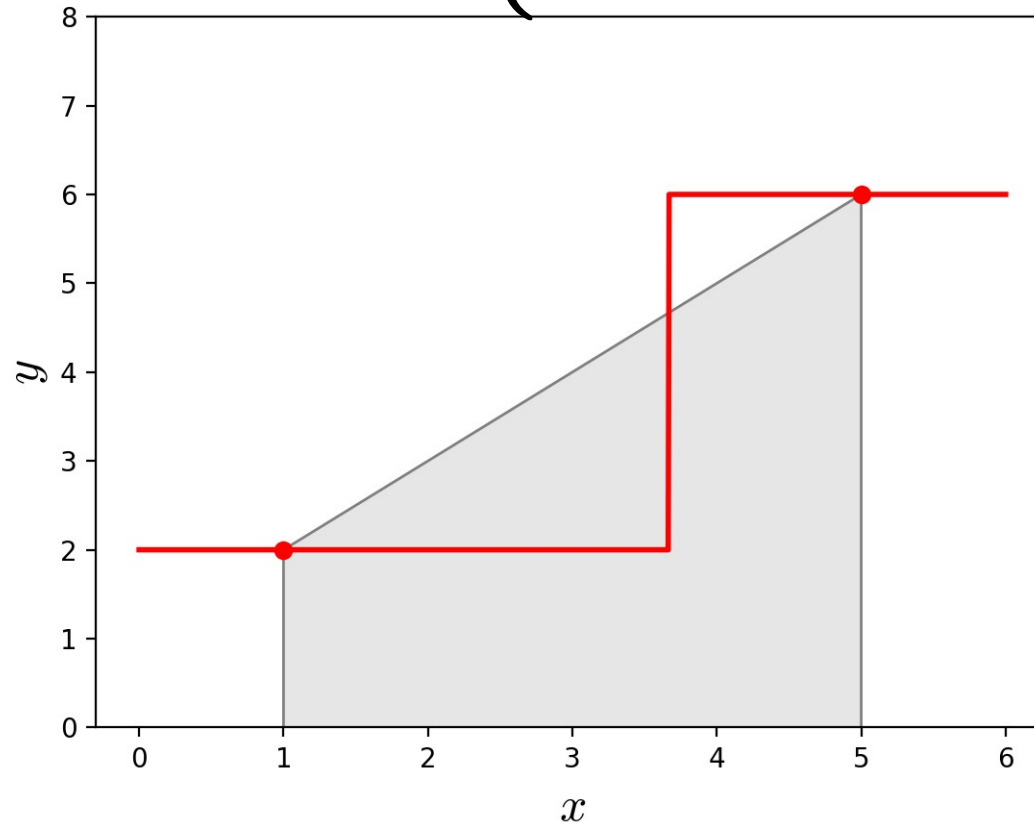


$$I[f] = 15.4336\dots \quad Q_0[f] = 8 \quad E[f] = 7.4336\dots$$

Bsp.: (9)

# Trapezregel

$$f(x) = \begin{cases} 2 & \text{if } x < \pi \\ 6 & \text{if } x \geq \pi \end{cases}$$

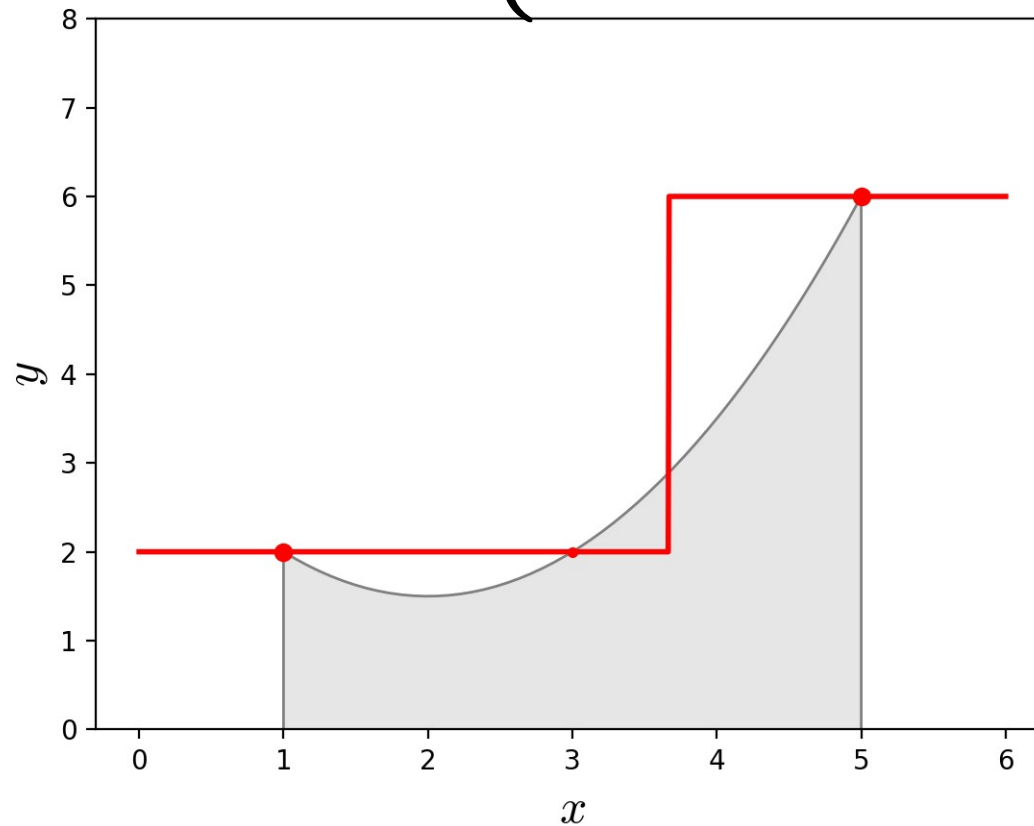


$$I[f] = 15.4336... \quad Q_1[f] = 16 \quad E[f] = 0.5664...$$

Bsp.: (9)

# Simpson-Regel

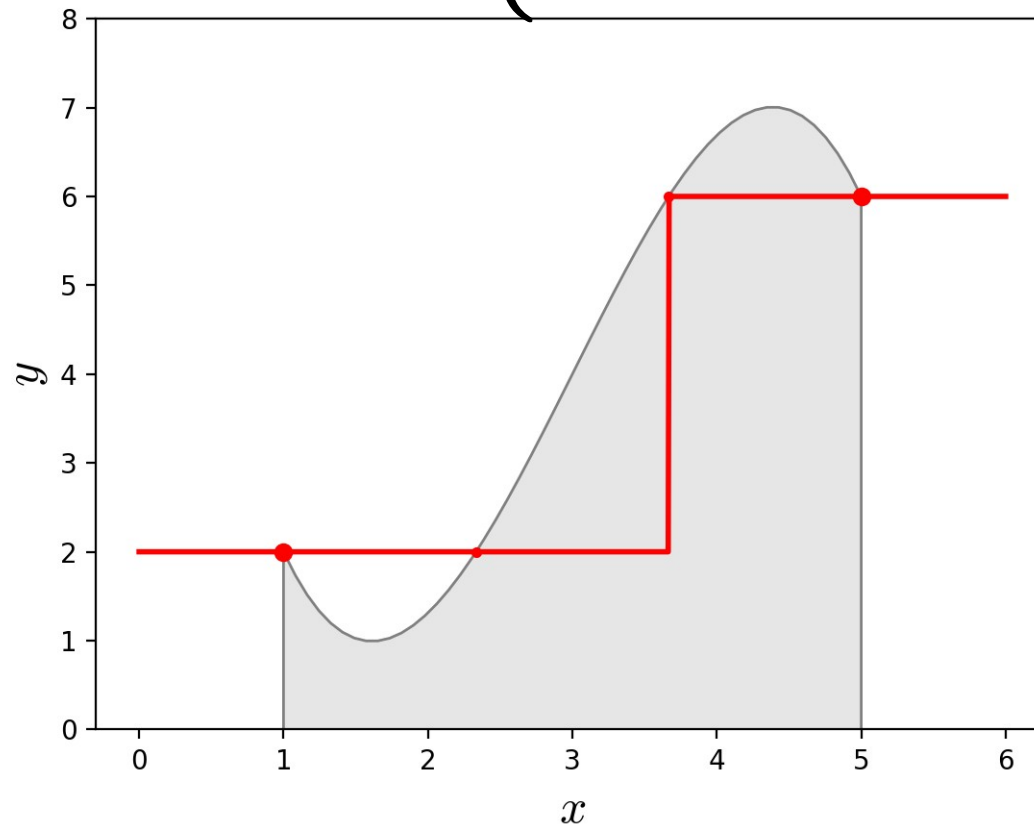
$$f(x) = \begin{cases} 2 & \text{if } x < \pi \\ 6 & \text{if } x \geq \pi \end{cases}$$



$$I[f] = 15.4336\dots \quad Q_2[f] = 10.6666\dots \quad E[f] = 4.7670\dots$$

# 3/8-Regel

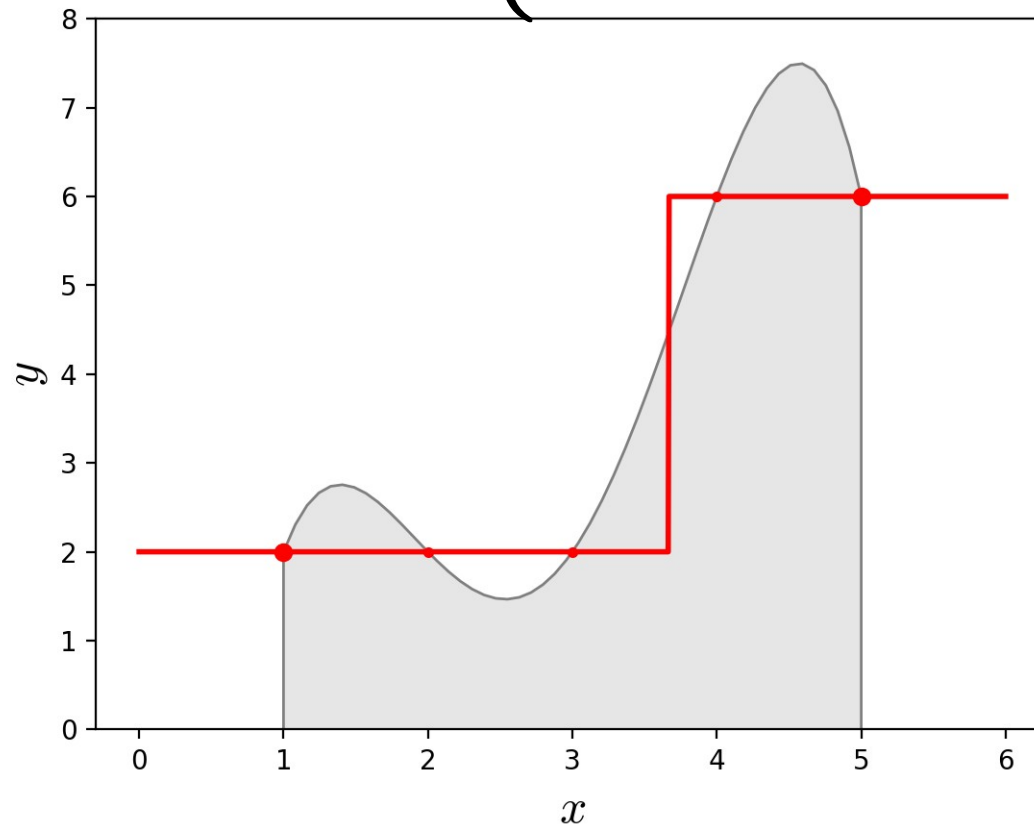
$$f(x) = \begin{cases} 2 & \text{if } x < \pi \\ 6 & \text{if } x \geq \pi \end{cases}$$



$$I[f] = 15.4336... \quad Q_3[f] = 16 \quad E[f] = 0.5664...$$

# Milne-Regel

$$f(x) = \begin{cases} 2 & \text{if } x < \pi \\ 6 & \text{if } x \geq \pi \end{cases}$$

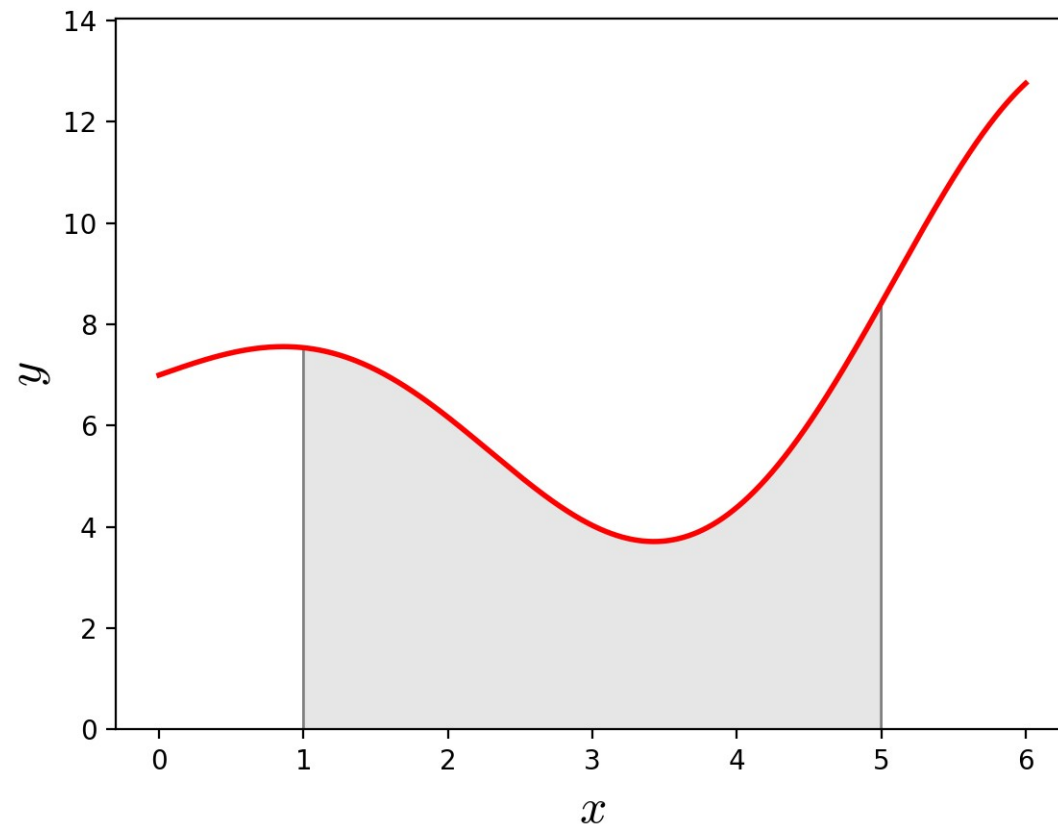


$$I[f] = 15.4336\dots \quad Q_4[f] = 14.9333\dots \quad E[f] = 0.5003\dots$$

Bsp.: (13)

# Summierte Quadratur-Regeln

$$f(x) = x \cos(x) + 7$$



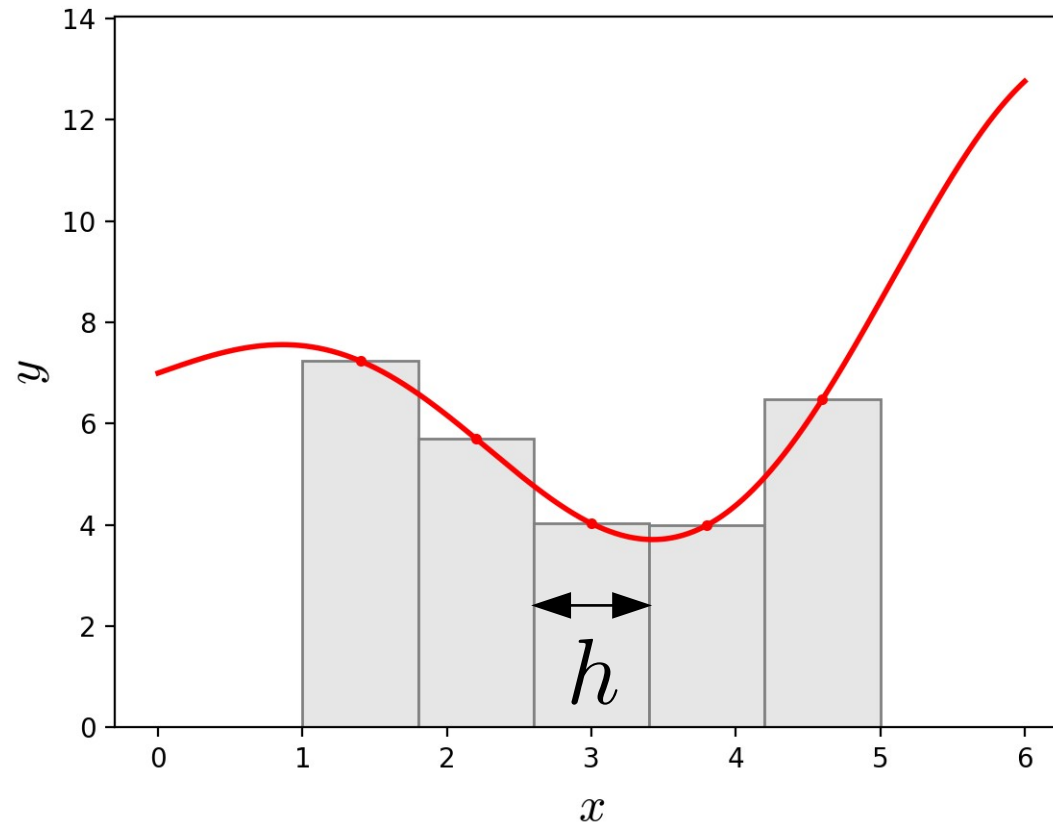
$$I[f] = \int_1^5 f(x) dx = 22.1073\dots$$



Bsp.: (13)

# Summierte Mittelpunktsregel

$$f(x) = x \cos(x) + 7$$

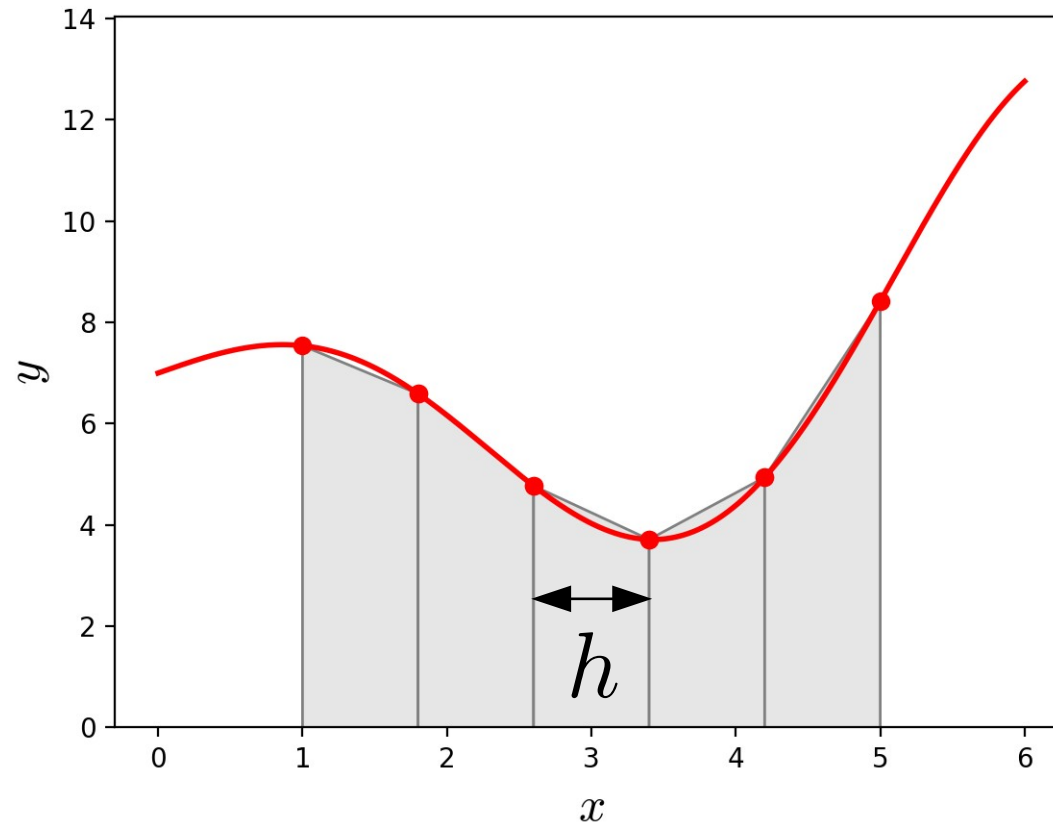


$$N = 5$$

Bsp.: (13)

# Summierte Trapezregel

$$f(x) = x \cos(x) + 7$$

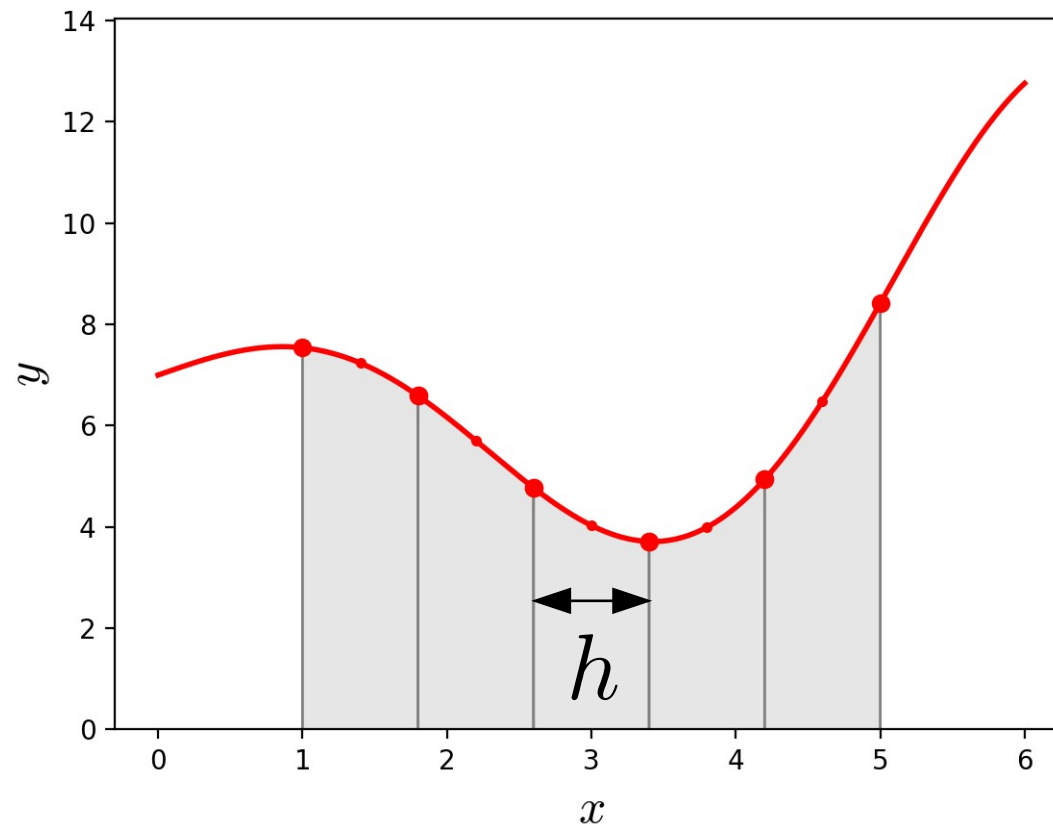


$$N = 5$$

Bsp.: (13)

# Summierte Simpson-Regel

$$f(x) = x \cos(x) + 7$$

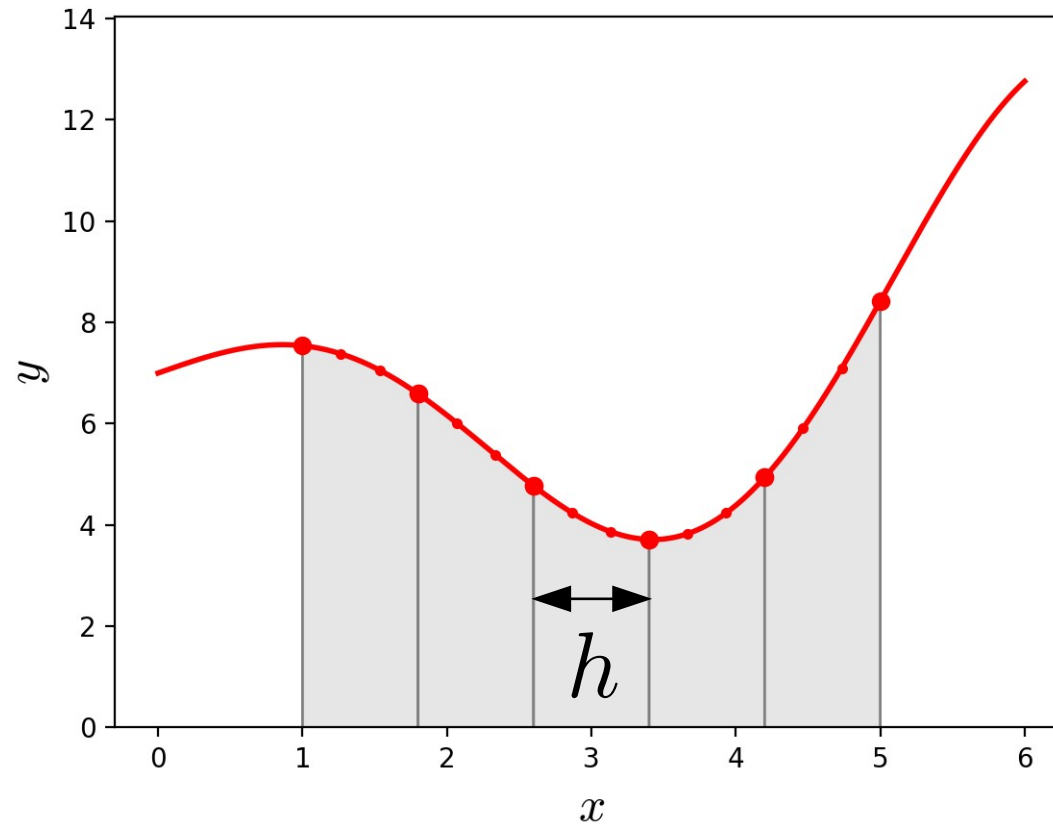


$$N = 5$$

Bsp.: (13)

# Summierte 3/8-Regel

$$f(x) = x \cos(x) + 7$$

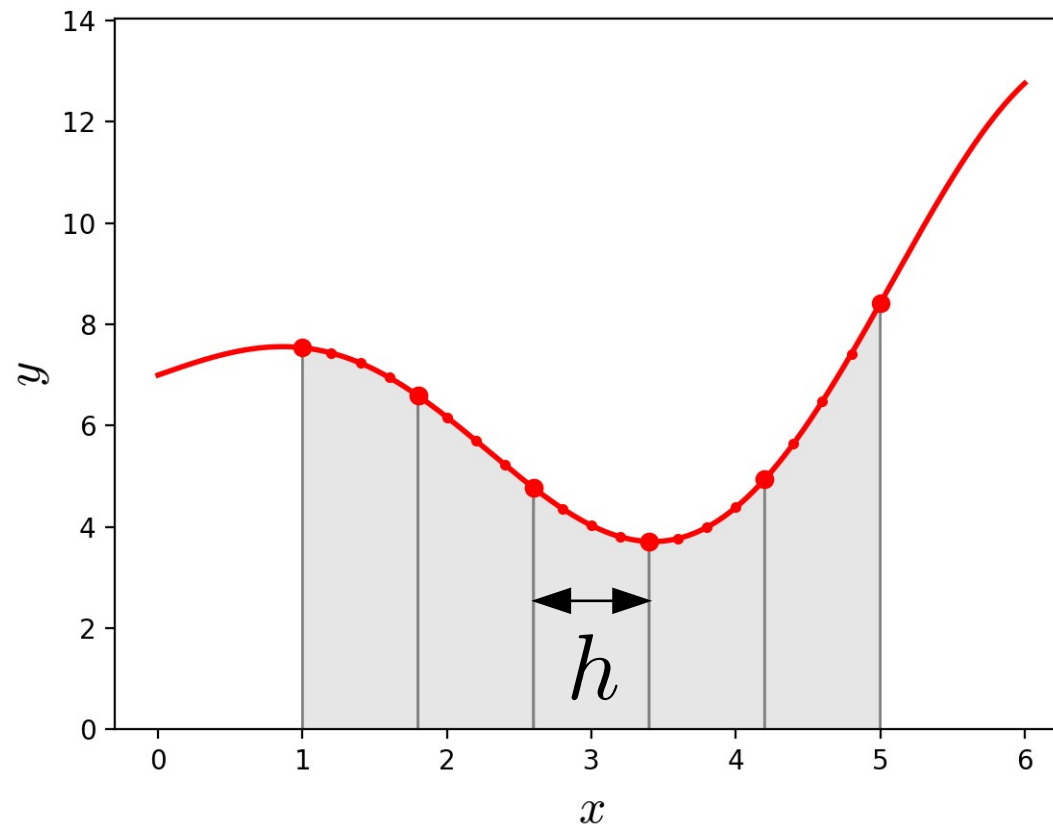


$$N = 5$$

Bsp.: (13)

# Summierte Milne-Regel

$$f(x) = x \cos(x) + 7$$



$$N = 5$$

# Summierte Quadratur-Regeln

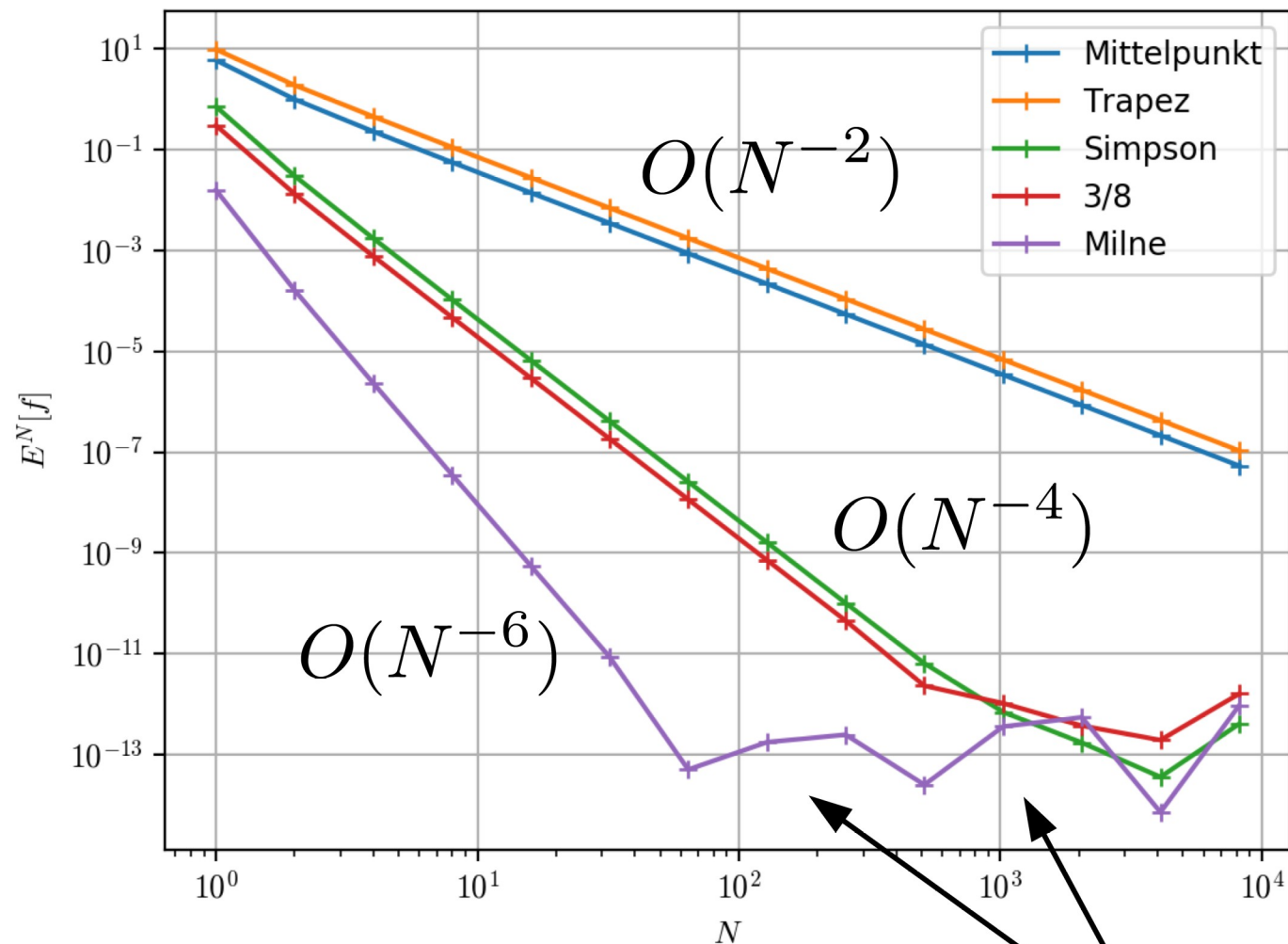
$$f(x) = x \cos(x) + 7$$

	Mittelpunkts	Trapez	Simpson	3/8	Milne	
$N$	$h$	$E_0^N[f]$	$E_1^N[f]$	$E_2^N[f]$	$E_3^N[f]$	$E_4^N[f]$
1	1.00E+00	5.99E+00	9.81E+00	7.21E-01	3.05E-01	1.59E-02
2	5.00E-01	1.00E+00	1.91E+00	3.02E-02	1.33E-02	1.62E-04
4	2.50E-01	2.30E-01	4.55E-01	1.74E-03	7.69E-04	2.32E-06
8	1.25E-01	5.64E-02	1.12E-01	1.06E-04	4.72E-05	3.54E-08
16	6.25E-02	1.40E-02	2.80E-02	6.61E-06	2.94E-06	5.50E-10
32	3.12E-02	3.50E-03	7.01E-03	4.13E-07	1.83E-07	8.62E-12
64	1.56E-02	8.76E-04	1.75E-03	2.58E-08	1.15E-08	4.97E-14
128	7.81E-03	2.19E-04	4.38E-04	1.61E-09	7.17E-10	1.74E-13

$$h = \frac{b - a}{N} \quad \dots$$

# Summierte Quadratur-Regeln

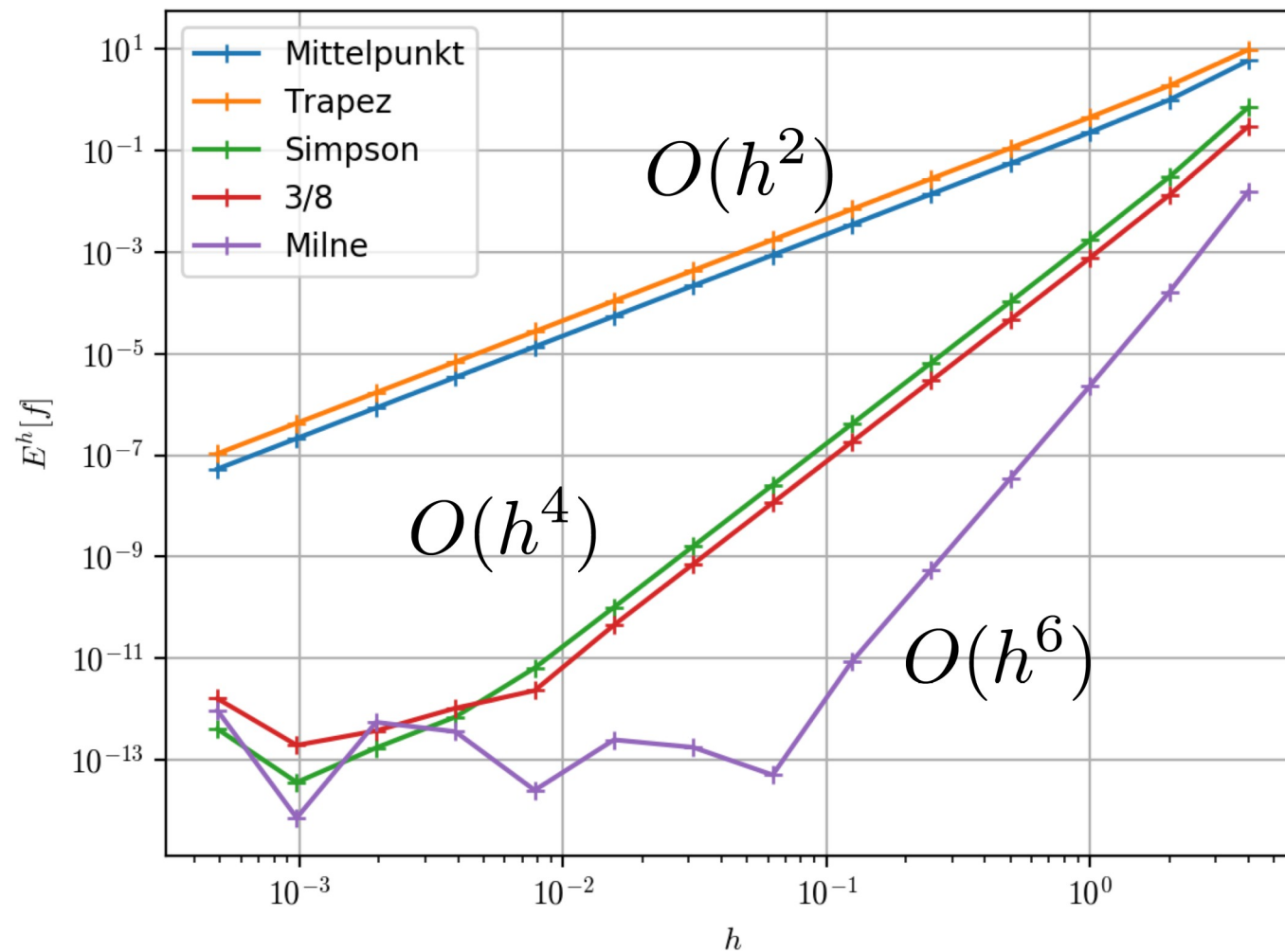
$$f(x) = x \cos(x) + 7$$



**Maschinengenauigkeit!**

# Summierte Quadratur-Regeln

$$f(x) = x \cos(x) + 7$$

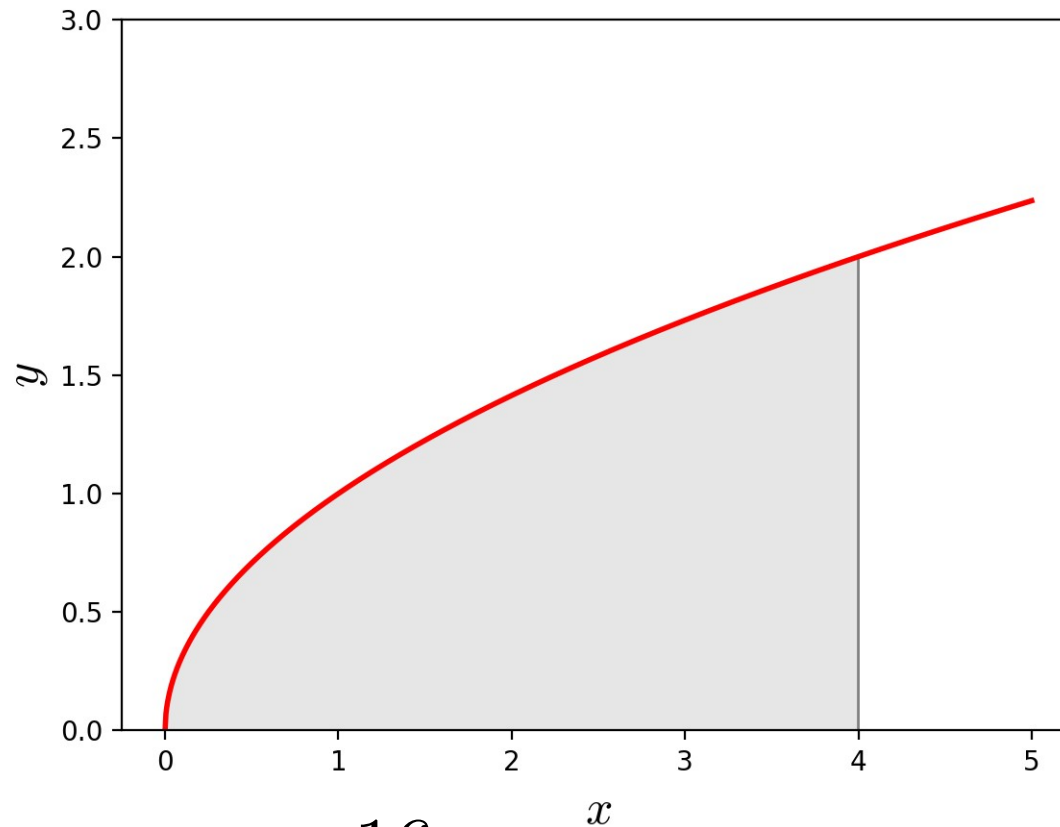




Bsp.: (13)

# Summierte Quadratur-Regeln

$$f(x) = \sqrt{x}$$

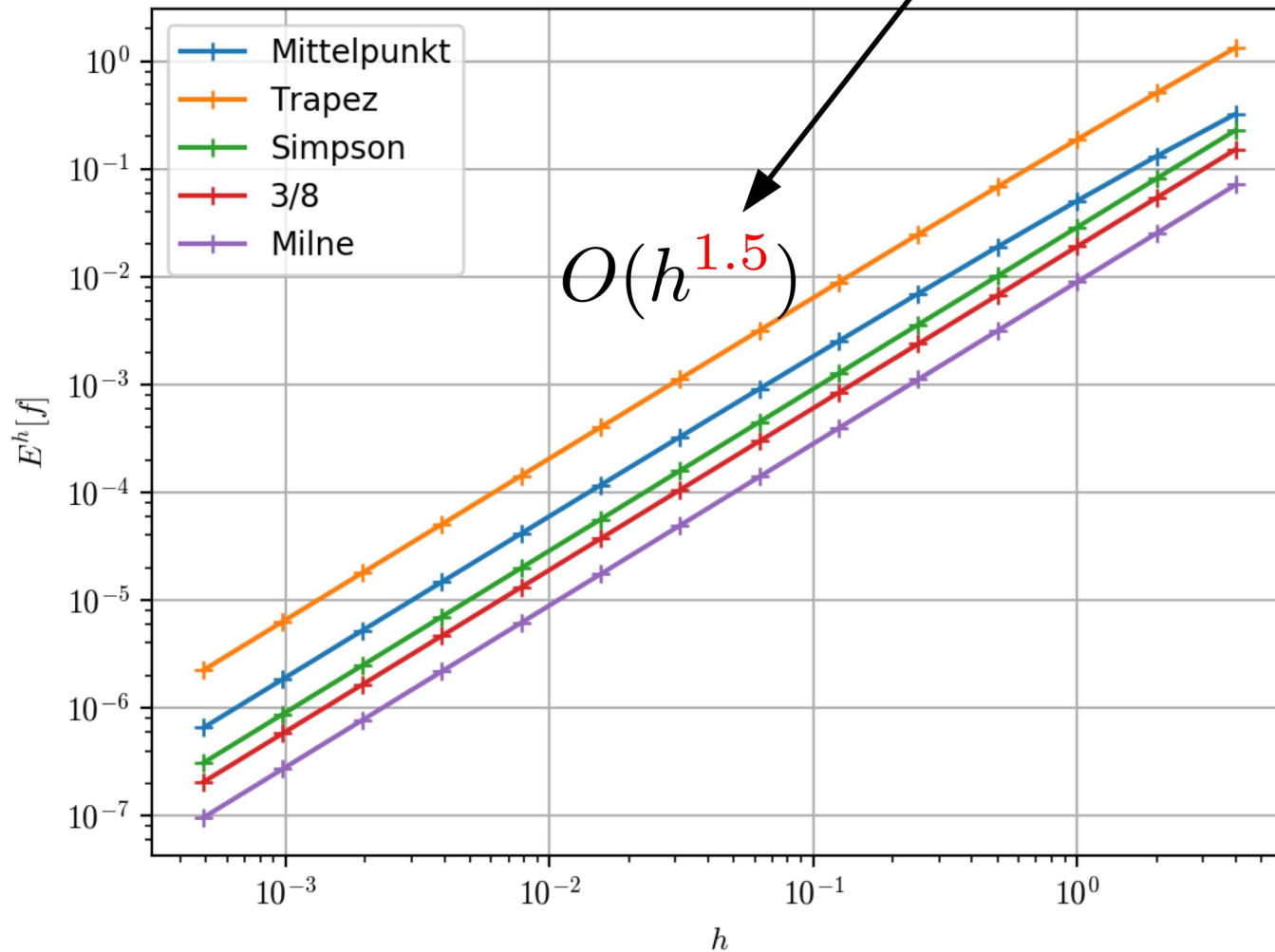


$$I[f] = \int_0^4 f(x) dx = \frac{16}{3}$$

# Summierte Quadratur-Regeln

$$f(x) = \sqrt{x}$$

$$? f'(x) = \frac{1}{2\sqrt{x}} \xrightarrow{x \rightarrow 0} \infty$$



...Glattheit!

## Gauss (Legendre) Quadratur

Die  $(n+1)$ -Punkte Gauss (Legendre) Quadratur (GLQ) auf dem RI  $[-1, 1]$  ist gegeben

durch:

$$G_n[f] = \sum_{j=0}^n w_j \cdot f(x_j)$$

wobei die Gauss-Punkte  $x_j$  die Nullstellen des  $(n+1)$ -ten Legendre-Polynoms  $P_{n+1}(x)$  und die Gewichte

$$w_j = \frac{2(1-x_j^2)}{((n+1)P_n(x_j))^2}, \quad j=0, 1, \dots, n$$

sind.

Sie hat den grösstmöglichen GG  $q=2n+1$

und damit Ordnung  $s=2n+2$ .

Bsp.: (14) 2-Punkte GLQ ( $n=1$ )

① Berechne  $P_{n+1}(x) = P_2(x)$  mit  
Rekursionsformel

$$P_2(x) = \frac{3}{2}x^2 - \frac{1}{2} = \frac{1}{2}(3x^2 - 1)$$

② Berechne Nullstellen von  $P_2(x)$

$$\frac{1}{2}(3x^2 - 1) \stackrel{!}{=} 0$$

$$x_{0,1} = \pm \frac{1}{\sqrt{3}}$$

③ Berechne Gewichte

$$\begin{aligned} w_0 &= \frac{2(1 - x_0^2)}{(2 \cdot P_1(x_0))^2} = \frac{2(1 - 1/3)}{(2 \cdot (-1/\sqrt{3}))^2} \\ &= \frac{\cancel{2} \cdot \overbrace{(1 - 1/3)}^{2/3}}{\cancel{2} \cdot \frac{1}{3}} = 1 \end{aligned}$$

$$w_1 = 1$$

Also  $G_1[f] = f\left(-\frac{1}{\sqrt{3}}\right) + f\left(\frac{1}{\sqrt{3}}\right)$

# Fehlerschätzer

Bsp.: (15)  $I[e^x] = \int_0^1 e^x dx = e - 1 = 1.71828\dots$

TR:  $Q_1[e^x] = \frac{1}{2}(e^0 + e^1) = 1.85914\dots$

SR:  $Q_2[e^x] = \frac{1}{6}(e^0 + 4 \cdot e^{1/2} + e^1) = 1.71886\dots$

STR:  $Q_1^2[e^x] = \frac{1/2}{2}(e^0 + 2 \cdot e^{1/2} + e^1) = 1.75393\dots$

Exakter Fehler :  $|Q_1[e^x] - I[e^x]| = 0.14085\dots$

Schätzung 1 :  $|Q_1[e^x] - Q_2[e^x]| = 0.14027\dots$

Schätzung 2 :  $|Q_1[e^x] - Q_1^2[e^x]| = 0.10520\dots$

Bsp.: (16) TR hat Ordnung  $s=2$  und

es ergibt sich für Zahlen aus

Bsp. (15) (Schätzung 2):

$$|Q_1[e^x] - I[e^x]| = 0.14085\dots$$

$$\approx \frac{2^2}{2^2-1} \cdot 0.10520\dots$$

$$= 0.1402\dots \quad \checkmark$$

$$|Q_1^2[e^x] - I[e^x]| = 0.0356\dots$$

$$\approx \frac{1}{2^2-1} \cdot 0.10520\dots$$

$$= 0.0350\dots \quad \checkmark$$

# Adaptive Quadratur Pseudo-Code

Adaptive Quadratur (~ Pseudo-MATLAB-Code)

function  $Q = \text{adapt\_quad}(f, a, b, \text{tol})$

if  $E < \text{tol}$

$Q = \text{quad}(f, a, b)$

else

$Q_1 = \text{adapt\_quad}(f, a, \frac{a+b}{2}, \frac{\text{tol}}{2})$

$Q_2 = \text{adapt\_quad}(f, \frac{a+b}{2}, b, \frac{\text{tol}}{2})$

$Q = Q_1 + Q_2$

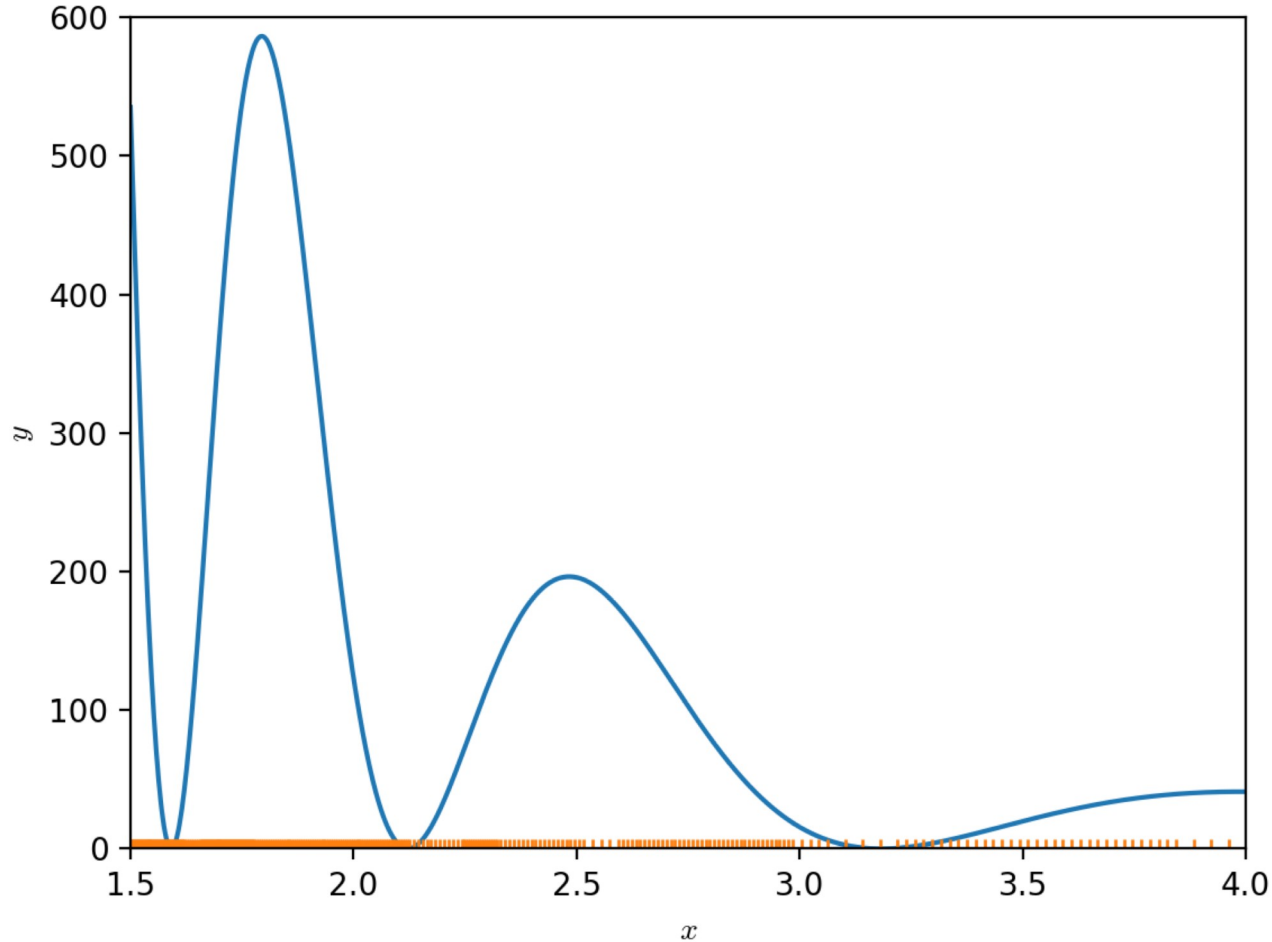
end

↑  
halbieren  
Fehler-Toleranz, da nun  
halbiertes Int.-Intervall!

Bsp.: (17)

# Adaptive Quadratur

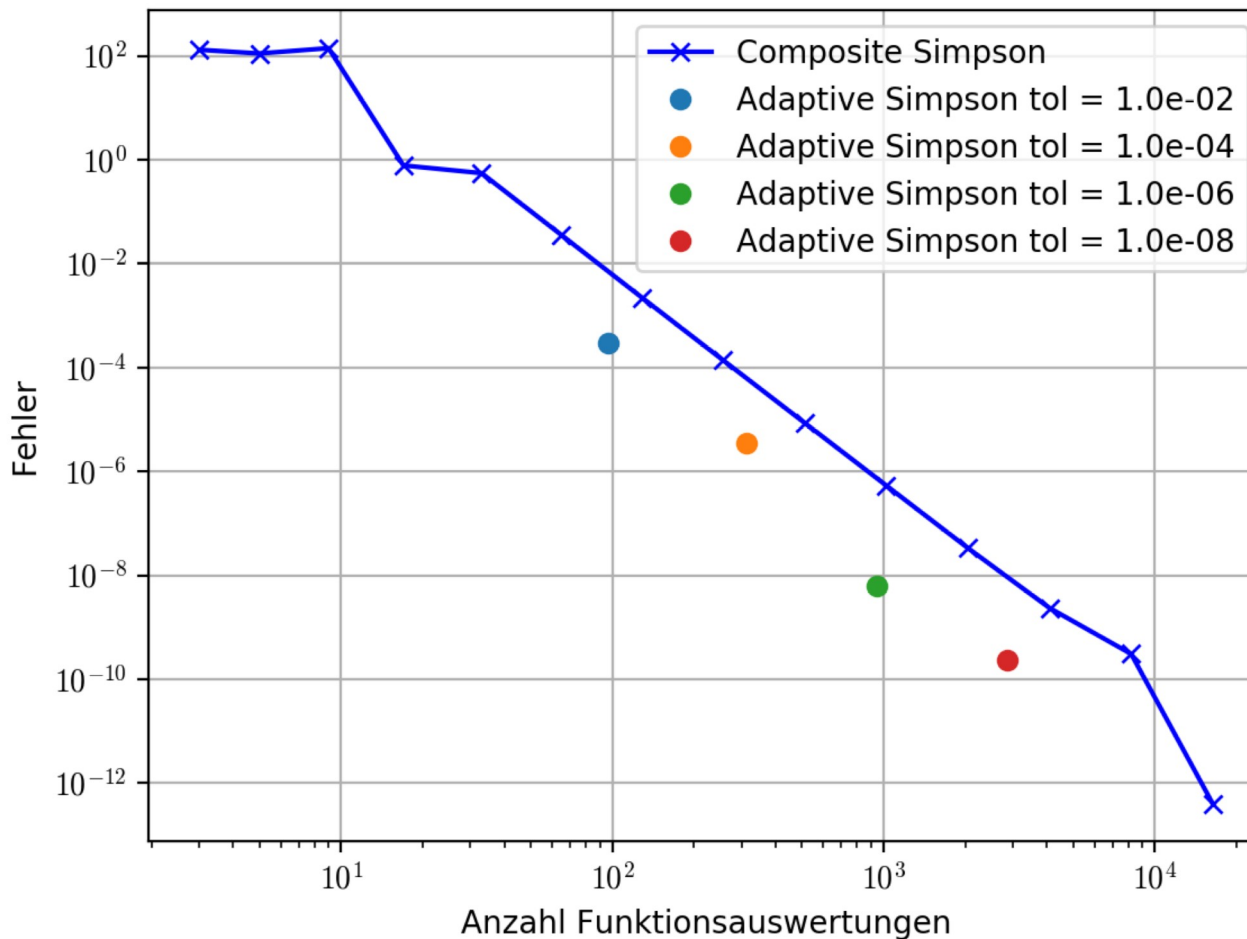
$$I[f] = \int_{3/2}^4 \left( \frac{200}{2x^3 - x^2} \right) (5 \sin(20/x))^2 dx$$





# Adaptive Quadratur

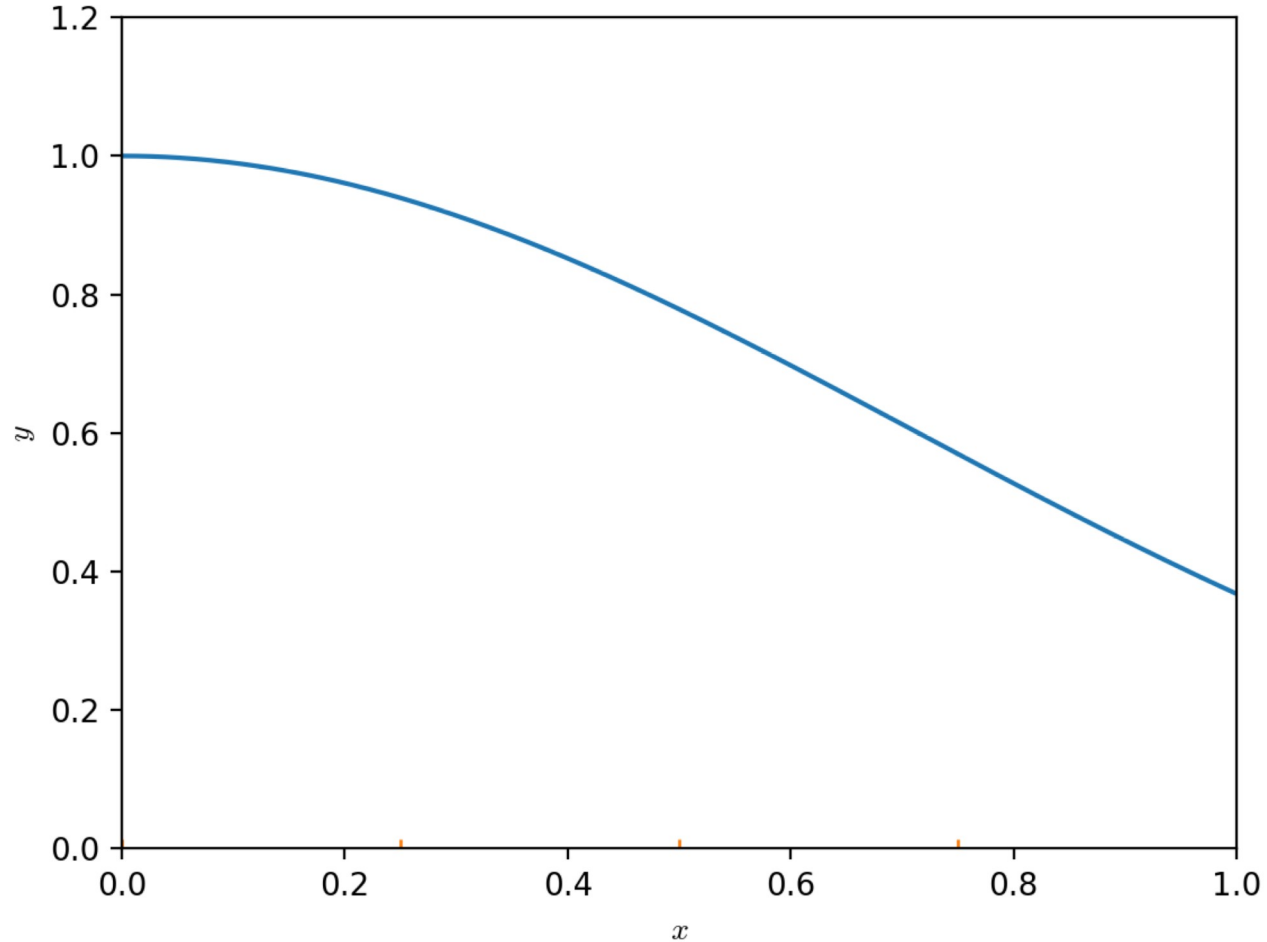
$$I[f] = \int_{3/2}^4 \left( \frac{200}{2x^3 - x^2} \right) (5 \sin(20/x))^2 dx$$



Bsp.: (17)

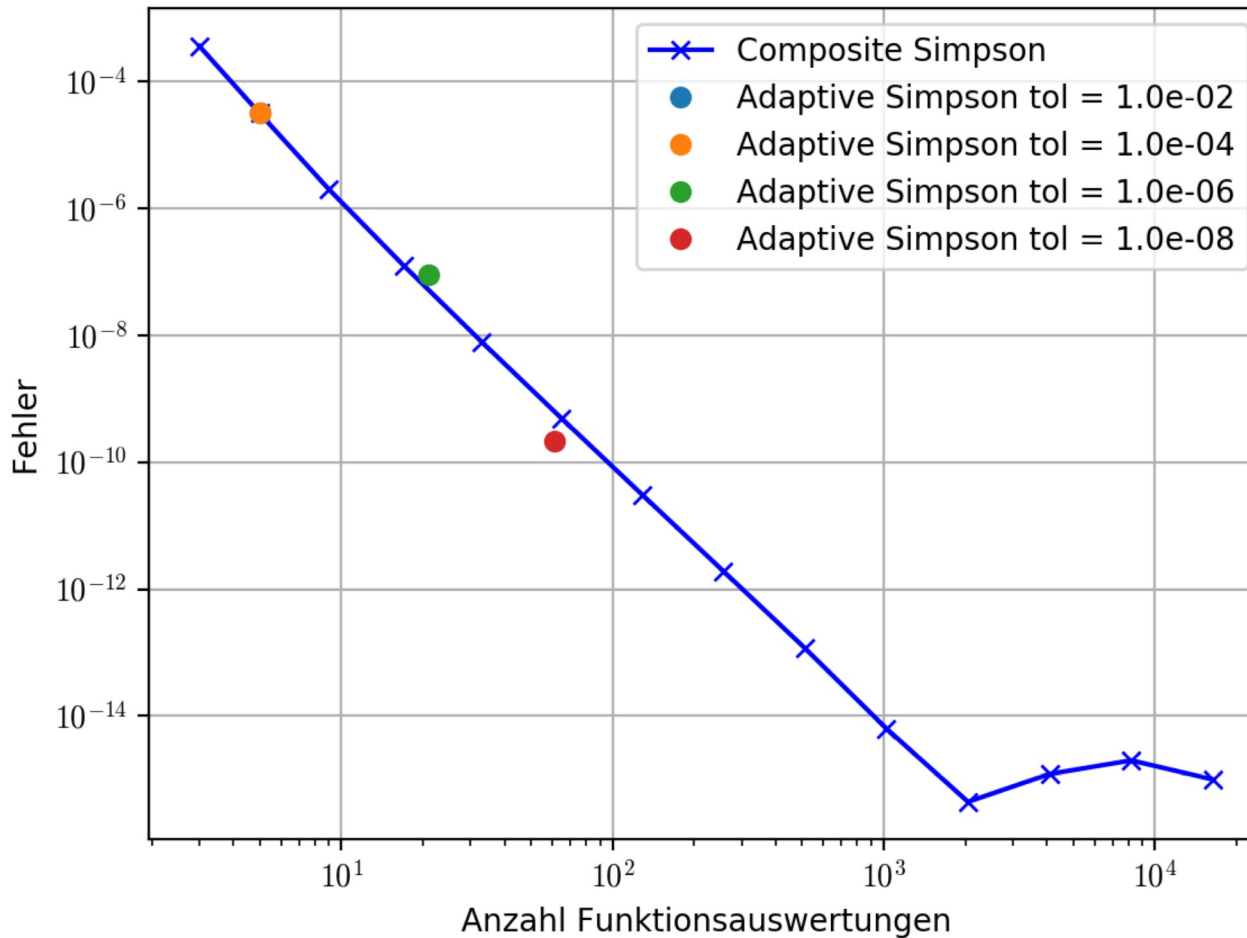
# Adaptive Quadratur

$$I[f] = \int_0^1 e^{-x^2} dx$$



# Adaptive Quadratur

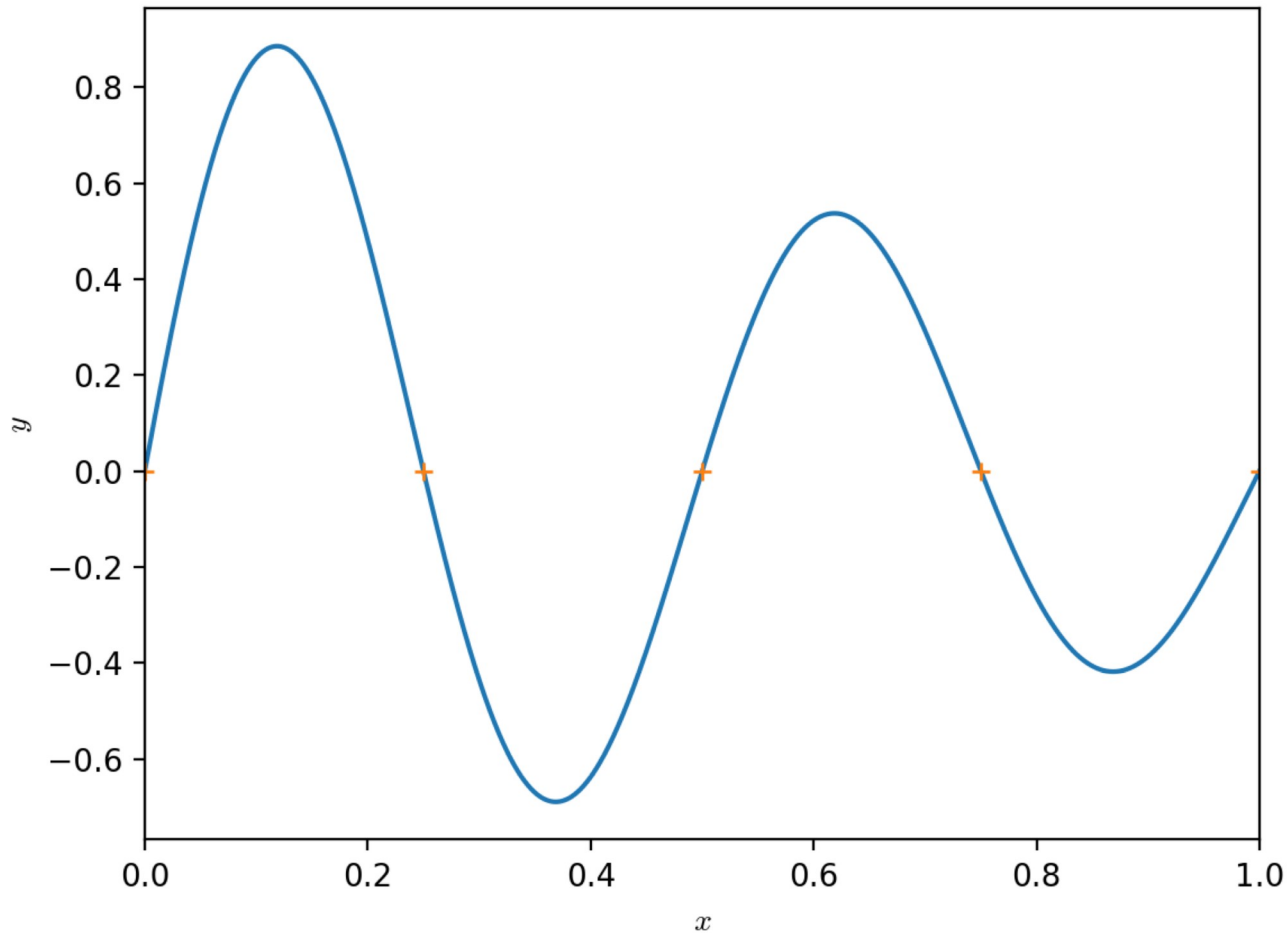
$$I[f] = \int_0^1 e^{-x^2} dx$$



Bsp.: (17)

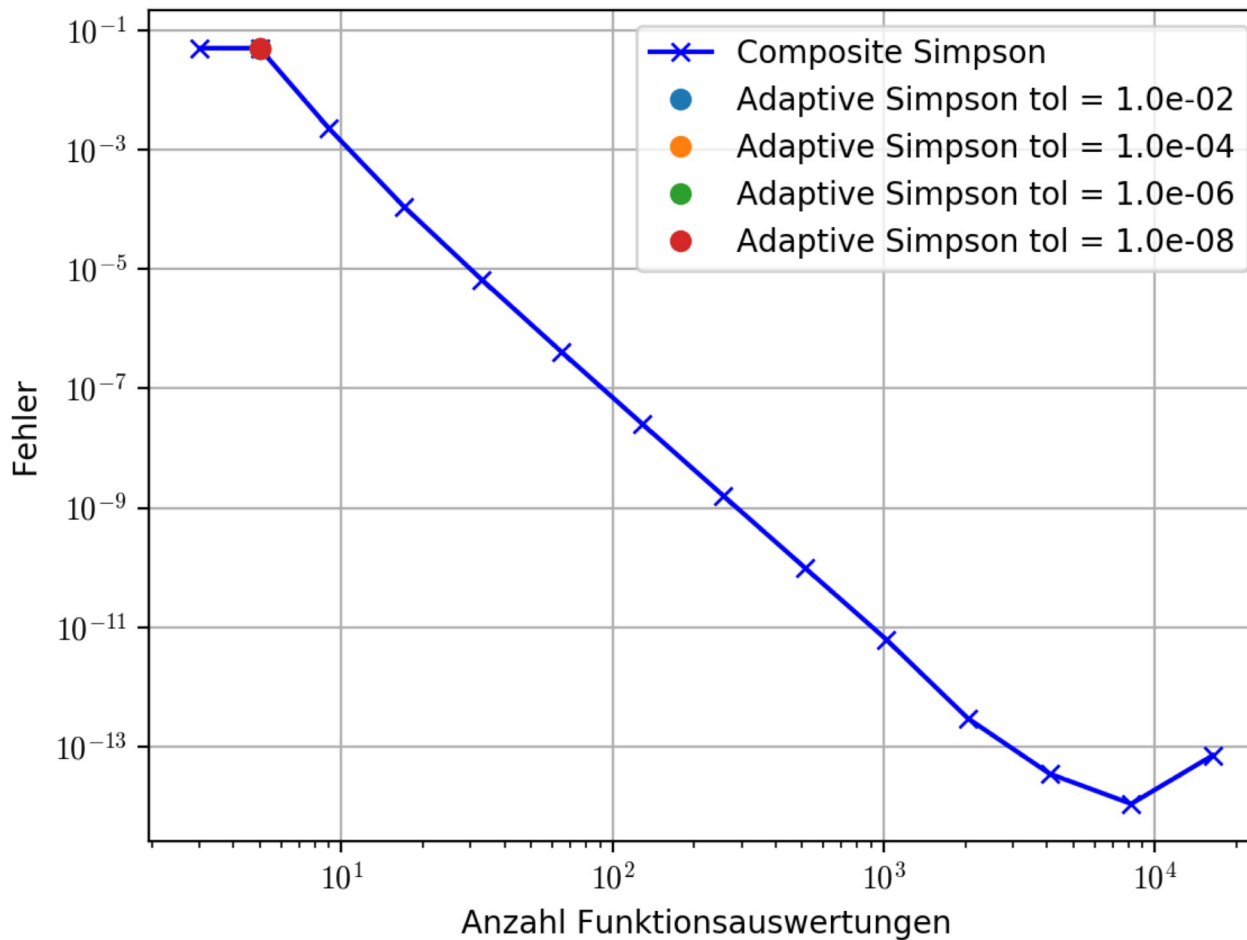
# Adaptive Quadratur

$$I[f] = \int_0^1 e^{-x} \sin(4\pi x) dx$$



# Adaptive Quadratur

$$I[f] = \int_0^1 e^{-x} \sin(4\pi x) dx$$



# I.8 Zwei-dimensionale Quadratur

$$I[F] = \int_a^b \int_c^d f(x,y) dx dy$$

$$\approx \sum_{i=0}^n w_i^x \cdot \int_c^d f(x_i, y) dy$$

Gewichte & Knoten in x-Koordinate

$$\approx \sum_{i=0}^n w_i^x \sum_{j=0}^m w_j^y \cdot f(x_i, y_j)$$

Gewichte & Knoten in y-Koord.

$$= \sum_{i=0}^n \sum_{j=0}^m w_i^x \cdot w_j^y \cdot f(x_i, y_j)$$

