Serie 14

Determinant

1. Compute the determinant of

$$B = \begin{pmatrix} 1 & 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 2 & 3 \\ 0 & 0 & 2 & 3 & 0 \\ 0 & 2 & 3 & 0 & 0 \\ 2 & 3 & 0 & 0 & 0 \end{pmatrix}$$

over $\mathbb{R}$ and over $\mathbb{F}_5$. Is B invertible?

2. Each of the following expressions defines a function D on the set of 3×3 matrices over the field of real numbers. In which of these cases is D a 3-linear function?

   (a) $D(A) = A_{11} + A_{22} + A_{33}$;

   (b) $D(A) = A_{11}^2 + 3A_{11}A_{22}$;

   (c) $D(A) = A_{11}A_{12}A_{33}$;

   (d) $D(A) = A_{13}A_{22}A_{32} + 5A_{12}A_{22}A_{32}$;

   (e) $D(A) = 0$;

   (f) $D(A) = 1$.

3. (a) Let K be a field, let $\lambda \in K$, and let $A \in M_{n\times n}(K)$. Show that:

   i. For B such that $A \xrightarrow{\lambda L_i \to L_i} B$, $\det B = \lambda \det A$;

   ii. For B such that $A \xrightarrow{L_i \leftrightarrow L_j} B$, $\det B = -\det A$;

   iii. For B such that $A \xrightarrow{\lambda L_i + L_j \to L_j} B$ with $i \neq j$, $\det B = \det A$.

   (b) The integers $2014, 1484, 3710$ and 6996 are all multiples of 1006. Show (without brute-force calculations) that also

$$\det \begin{pmatrix} 2 & 1 & 3 & 6 \\ 0 & 4 & 7 & 9 \\ 1 & 8 & 1 & 9 \\ 4 & 4 & 0 & 6 \end{pmatrix}$$

   is a multiple of 106.

4. Compute the determinants of the following matrices

$$A = \begin{pmatrix} 0 & 0 & 0 & 1 \\ -1 & 2 & 0 & 1 \\ 1 & 2 & -3 & 1 \\ 0 & -4 & 2 & -1 \end{pmatrix}, \quad B = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 \\ -1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & -1 & 0 \\ 0 & 0 & -1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 \end{pmatrix},$$

$$C = \begin{pmatrix} 2 & -3 & 5 & 1 & 4 \\ 2 & -3 & 1 & -6 & 18 \\ 4 & -3 & 9 & 6 & 10 \\ -2 & 4 & -6 & -1 & -1 \\ -6 & 11 & -23 & -14 & 9 \end{pmatrix}.$$

5. Let $A_n \in \mathrm{M}(n \times n, \mathbb{R})$ be the matrix

$$\begin{pmatrix} 2 & 1 & 0 & \dots & 0 \\ 1 & 2 & 1 & \dots & 0 \\ 0 & \ddots & \ddots & \ddots & \vdots \\ \vdots & & 1 & 2 & 1 \\ 0 & \dots & 0 & 1 & 2 \end{pmatrix}$$

Prove that

$$\det(A_n) = n + 1.$$

6. Let K be a subfield of the complex numbers and n a positive integer. Let $j_1, \dots, j_n$ and $k_1, \dots, k_n$ be positive integers not exceeding n. For an $n \times n$ matrix A over K define

$$D(A) = A(j_1, k_1)A(j_2, k_2) \cdots A(j_n, k_n).$$

Prove that D is n-linear if and only if the integers $j_1, \dots, j_n$ are distinct.

Single Choice. In each exercise, exactly one answer is correct.

1. For which $x \in \mathbb{R}$ do we have $\det \begin{pmatrix} 1 & 1 & 1 \\ 1 & x & 1 \\ 1 & 1 & x \end{pmatrix} = 1$?

   (a) $x = -2$

   (b) $x = 2$

   (c) $x = -1$

   (d) $x = 1$

2. Let K be a field and $\mathrm{Mat}_{n\times n}(K)$ the vector space of $n \times n$-Matrizen over K. Which assertion is wrong in general?

   (a) A marix $A \in \mathrm{Mat}_{n\times n}(K)$ over K is invertible if and only if $\det(A) \neq 0$.

   (b) The determinant of an upper triangular matrix only depends on its diagonal entries.

   (c) For every $n \geqslant 0$ the determinant is a linear map $\mathrm{Mat}_{n\times n}(K) \to K$.

   (d) For every $n > 0$ the determinant map $Mat_{n\times n}(K) \to K$ is surjektive.

3. In general, which operation changes the determinant?

   (a) Exchanging two rows.

   (b) Adding the multiple of one row to another

   (c) Transpose.

   (d) Replacement by similar matrix.

Multiple Choice Fragen.

1. Which of the following assertions are correct for arbitrary $A, B \in M_{n\times n}(\mathbb{R})$ with $n \geqslant 2$?

   (a) We have $\det(AB) = \det(A)\det(B)$.

   (b) From $\det(A) \neq 0$ it follows that the column vectors $a_1, \cdots, a_n$ of A are linearly independent.

   (c) Es gilt $\det(AB) = \det(BA)$.

   (d) For every non-zero real number λ we have $\det(\lambda A) = \lambda \det(A)$.

   (e) We have $\det(A + B) = \det(A) + \det(B)$.

2. Let $\det \begin{pmatrix} a & b \\ c & d \end{pmatrix} = 4$. Which of the following statements are true?

   (a) $\det \begin{pmatrix} 2a & 2b \\ 2c & 2d \end{pmatrix} = 8$.

   (b) $\det \begin{pmatrix} a & b \\ c - a & d - b \end{pmatrix} = 4$.

   (c) $\det \begin{pmatrix} a & b \\ c + 2a & d + 2b \end{pmatrix} = 4$.

   (d) $\det \begin{pmatrix} a & b \\ 3c & 3d \end{pmatrix} = 12$.