Prof. M. Einsiedler
Prof. P. Biran

Serie 15

Determinant

1. Let K be a commutative ring with identity. If A is a 2×2 matrix over K, the classical adjoint of A is the 2×2 matrix adj A defined by

$$\operatorname{adj} A = \begin{pmatrix} A_{22} & -A_{12} \\ -A_{21} & A_{11} \end{pmatrix}$$

If det denotes the unique determinant function on 2×2 matrices over K, show that

(a) $(\operatorname{adj} A)A = A(\operatorname{adj} A) = (\det A)I$;

(b) $\det(\operatorname{adj} A) = \det(A)$;

(c) $\operatorname{adj}(A^t) = (\operatorname{adj} A)^t$.

(A^t denotes the transpose of A.)

2. (a) List explicitly the 24 permutations of degree 4, state which are odd and which are even, and use this to give the complete Leibniz formula

$$\det(A) = \sum_\sigma (\operatorname{sgn}\sigma) A(1, \sigma 1) \cdots A(n, \sigma n)$$

for the determinant of a 4×4 matrix. Notice that for $n \geqslant 4$ it is not sufficient to compute a combination of the diagonals of a matrix to obtain its determinant.

(b) For a general $n \in \mathbb{N}_{\geqslant 1}$, how many *even* permutations are there in S_n?

3. An $n \times n$ matrix A is called triangular if $A_{ij} = 0$ whenever $i > j$ or if $A_{ij} = 0$ whenever $i < j$. Prove that the determinant of a triangular matrix is the product $A_{11}A_{22} \cdots A_{nn}$ of its diagonal entries.

4. Let $n \in \mathbb{N}_{\geqslant 2}$. Show that

$$\det \begin{pmatrix} 1 & x_1 & x_1^2 & \cdots & x_1^{n-1} \\ 1 & x_2 & x_2^2 & \cdots & x_2^{n-1} \\ \vdots & \vdots & \vdots & & \vdots \\ 1 & x_n & x_n^2 & \cdots & x_n^{n-1} \end{pmatrix} = \prod_{1 \leqslant i < j \leqslant n} (x_j - x_i).$$

Remark. Products of the sort are called a *Vandermonde determinants* and the above matrix is called a *Vandermonde matrix.*

5. Let K be a field and let $A, B, C, D \in M_{n\times n}(K)$. Assume that A and C commute and that $\det A \neq 0$. Show that

$$\det\left(\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right) = \det(A \cdot D - C \cdot B)$$

Hint. Consider the matrix

$$\left(\begin{array}{c|c} I_n & O_n \\ \hline -C & A \end{array}\right).$$

6. Prove the following proposition using the Leibniz formula for determinants:

Proposition. *Let K be a field, and let $A, B \in M_{n\times n}(K)$. Then*

$$\det(AB) = \det(A) \cdot \det(B).$$

Hint. Denote $\{\mathbf{e}_i\}_{i=1}^n$ the standard basis of K^n and write the matrix B as a list of column blocks:

$$B = \left(\sum_{s_1=1}^n B(s_1, 1)\mathbf{e}_{s_1} \;\middle|\; \cdots \;\middle|\; \sum_{s_n=1}^n B(s_n, n)\mathbf{e}_{s_n} \right).$$

You will also need to prove the following lemma

Lemma. For any $A \in M_{n\times n}(K)$ and any $\sigma \in S_n$, we have

$$\det\left(A \cdot \mathbf{e}_{\sigma(1)} \;\middle|\; \cdots \;\middle|\; A \cdot \mathbf{e}_{\sigma(n)} \right) = \mathrm{sgn}(\sigma) \det(A).$$