Serie 16

Characteristic polynomials, Eigenvectors, Eigenvalues

1. Consider the matrix $A = \begin{pmatrix} 3 & 0 & -2 \\ 2 & 0 & -2 \\ 0 & 1 & 1 \end{pmatrix}$ over $\mathbb{R}$.

   (a) Determine the characteristic polynomial of A.

   (b) Determine the eigenvalues of A.

   (c) The *geometric multiplicity* of an eigenvector is the dimension of its eigenspace. The *arithmetic multiplicity* of an eigenvector is the multiplicity of this eigenvector as a zero of the characteristic polynomial. Determine the arithmetic and geometric multiplicity of all eigenvalues.

2. Compute the characterisitc polynomial, the eigenvalues and eigenvektors of the following matrices over $\mathbb{Q}$ and check if they are diagonalizable.

   (a) $A := \begin{pmatrix} 1 & -1 \\ 2 & 4 \end{pmatrix}$

   (b) $B := \begin{pmatrix} 2 & 2 & 3 \\ 1 & 2 & 1 \\ 2 & -2 & 1 \end{pmatrix}$

   (c) $C := \begin{pmatrix} -4 & -3 & -1 & -7 \\ -3 & -1 & -1 & -4 \\ 6 & 4 & 3 & 8 \\ 3 & 3 & 1 & 6 \end{pmatrix}$

3. For an arbitrary invertible $n \times n$-matrix A, write the characteristic polynomial of A^{-1} in terms of the characteristic polynomial of A.

4. Let K^∞ be the vectorspace of all infinite sequences in K, and let K_0^∞ be the subspace of all sequences where all but finitily many elements are 0.

   (a) Determine all eigenvalues and eigenvectors of the endomorphism

   $$T : K^\infty \to K^\infty,\ (x_0, x_1, x_2, \ldots) \mapsto (x_1, x_2, x_3, \ldots)\,.$$

   (b) Do the same for the induced endomorphism $K_0^\infty \to K_0^\infty$.

(c) Construct an endomorphism of K_0^∞ with eigenvalues $0, 1, 2, 3, \ldots$

(d) Construct an endomorphism of K_0^∞ which has no Eigenvalues.

5. Let A be a nilpotent $n \times n$-matrix. This means that there exists $m \geqslant 1$ with $A^m = O$. Show that the only possible eigenvalue of A is 0. When exactly is 0 an eigenvalue of A?

6. Let V be a K-vectorspace and let $F, G \in \mathrm{End}(V)$. Show:

   (a) If $v \in V$ is an eigenvektor of $F \circ G$ with eigenvalue λ and $G(v) \neq 0$, then $G(v)$ is an eigenvector of $G \circ F$ with eigenvalue λ.

   (b) If V is finite-dimensional, the endomorphisms $F \circ G$ and $G \circ F$ have the same eigenvalues.

   (c) Find a counterexample to (b) if V is not finite-dimensional.