Serie 17

Eigenvectors, Eigenvalues

1. In each of the following cases, let T_i be the endomorphism of $\mathbb{R}^2$ which is represented by the matrix A_i in the standard ordered basis for $\mathbb{R}^2$, and let U_i be the endomorphism of $\mathbb{C}^2$ represented by A_i in the standard ordered basis. Find the characteristic polynomial for T_i and that for U_i, find the eigenvalues of each endomorphism, and for each such eigenvalue find a basis for the corresponding space of eigenvectors.

$$A_1 = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \quad A_2 = \begin{pmatrix} 2 & 3 \\ -1 & 1 \end{pmatrix}, \quad A_3 = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}.$$

2. Let K be a field and let V be a finite-dimensional vector space over K. Suppose that $T \in \operatorname{End}(V)$ is invertible. Prove that $\operatorname{Eig}_T(\lambda) = \operatorname{Eig}_{T^{-1}}(1/\lambda)$ for every $\lambda \in K^*$.

3. Consider the space $C^\infty(\mathbb{R})$ of smooth functions over $\mathbb{R}$ and the map

$$\begin{array}{rccc} T: & C^\infty(\mathbb{R}) & \to & C^\infty(\mathbb{R}) \\ & f & \mapsto & f' \end{array}$$

   Find the eigenvalues and the corresponding eigenfunctions (this is a synonym for eigenvectors when working on a space whose elements are functions) of T.

4. Let $K = \mathbb{R}$, show that K^∞ does not admit any countable basis.

   Hint: Use the fact that pairwise distinct eigenvalues correspond to a set of linearly independent eigenvectors.

5. (a) Let f be an endomorphism of a finite-dimensional vector space V, and let $V = V_1 \oplus \ldots \oplus V_r$ with f-invariant subpaces V_i. Show, that the arithmetic, resp. geometric multiplicities of an eigenvalue $\lambda \in K$ of f is equal to the sum of the arithmetic, resp. geometric multiplies of λ as an eigenvalue of the endomorphisms $f|_{V_i}$ of V_i.

   (b) Deduce that f is diagonalizable if and only if $f|_{V_i}$ is diagonalizable for every i.

(c) Let f and g be endomorphisms for the same finite dimensional vector space V. Show that f and g are *simultaneously diagonalizable* (meaning that there exists a basis of eigenvectors of f which are all also eigenvectors of g) if and only if they commute and are diagonalizable.

Hint: To prove the the backward direction, first show that each eigenspace of f is g-invariant, i.e. that g maps eigenvectors of f to eigenvectors of f *in the same eigenspace.*

6. Let K be a field and let V be an n-dimensional vector space over K ($n > 0$).

(a) Let T be a diagonalizable endomorphism of V with (not necessarily distinct) eigenvalues λ_i for $1 \leqslant i \leqslant n$. Show that

$$\mathrm{Tr}(T) = \sum_{i=1}^{n} \lambda_i \quad \text{and that} \quad \det(T) = \prod_{i=1}^{n} \lambda_i.$$

For $0 \leqslant k \leqslant n$, let c_k be the coefficient of x^k in the characteristic polynomial of T. Give a formula for c_k in terms of the eigenvalues of T.

(b) Let $B \in M_{2\times 2}(\mathbb{R})$ be diagonalizable with $\mathrm{Tr}(B) = 0$. Show that $\det(B) \leqslant 0$.