D-MATH | Lineare Algebra II | FS 2023

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Serie 18

Diagonalizabiltiy, Cayley-Hamilton

1. Let K be a field, let $n \geqslant 2$, and let

$$A = \begin{pmatrix} 0 & 0 & \cdots & 0 & -c_0 \\ 1 & 0 & \cdots & 0 & -c_1 \\ 0 & 1 & \cdots & 0 & -c_2 \\ 0 & 0 & \cdots & 0 & -c_3 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & 1 & -c_{n-1} \end{pmatrix} \in M_{n\times n}(K).$$

Prove that

$$\mathrm{char}_A(X) = (-1)^n (X^n + c_{n-1}X^{n-1} + \cdots + c_0).$$

Hint: Use induction.

2. Let A be a $n \times n$-matrix of rank r. Show that the degree of the minimal polynomial of A is smaller or equal than $r+1$.

3. Prove that every 2×2 invertible real matrix belongs to one of the following categories:

   - It is diagonalizable;
   - it is trigonalizable with algebraic multiplicity 2 and geometric multiplicity 1;
   - one can find a basis such that the matrix representation in that basis is

$$\begin{pmatrix} a & b \\ -b & a \end{pmatrix} \quad \text{with } b \neq 0.$$

4. Let K be a field, $A \in M_{n\times n}(K)$, and $p \in K[X]$ be a non-trivial polynomial such that $p(A) = 0$. Show that every eigenvalue of A is a root of p.

   Hint: For an eigenvector v of A and a polynomial q over K, state and prove the relationship between v and $q(A)\cdot v$.

5. (a) Let A be a $n\times n$-matrix. Prove that the subspace $\langle I_n, A, A^2, \dots\rangle$ of $M_{n\times n}(K)$ has dimension $\leqslant n$.

(b) Let $A := \begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \\ 3 & 1 & 2 \end{pmatrix}$. Find a polynomial $p(X)$ with $p(A) = A^{-1}$.

6. Prove or disprove: There exists a real $n \times n$-matrix A satisfying

$$A^2 + 2A + 5I_n = 0$$

if and only if n is even.