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Serie 19

Scalar products, bilinear forms

1. For which values of $a \in \mathbb{R}$ does the expression

$$\langle x, y \rangle := x_1 y_1 + a x_1 y_2 + a x_2 y_1 + 7 x_2 y_2,$$

where $x = \binom{x_1}{x_2}$ and $y = \binom{y_1}{y_2}$, define an inner product on $\mathbb{R}^2$?

2. Let V be the vector space of real polynomials of degree at most n.

   (a) Show that the expression

$$\langle p, q \rangle := \int_0^\infty p(t) q(t) e^{-t} \, dt$$

   defines an inner product on V.

   (b) Find the matrix of the inner product with respect to the basis $1, x, \ldots, x^n$.

3. Let $V = \mathbb{R}^2$ endowed with the standard scalar product, and for $i = 1, 2$, let $v_i \in V \smallsetminus \{0\}$. Show that the formula

$$\langle v_1, v_2 \rangle = ||v_1|| \, ||v_2|| \cos(\widehat{v_1, v_2}),$$

defining the cosinus of an angle is rotation-invariant. In other words show that, for any rotation of the plane $R : V \to V$, we have

$$\cos(\widehat{v_1, v_2}) = \cos(\widehat{Rv_1, Rv_2}),$$

which is what we would expect from a good definition of the angle between 2 vectors.

4. Let $A \in \mathrm{Mat}_{n \times n}(\mathbb{R})$. Show that:

   (a) The matrix $A^T A$ is symmetric.

   (b) The matrix $A^T A$ is positive-definite if and only if A is invertible.

   (c) It holds that $\mathrm{Rang}(A^T A) = \mathrm{Rang}(A)$.

5. (a) Let $\|\cdot\|$ be a norm on the $\mathbb{R}$-vector space V. Show that the norm is induced by an inner product $\langle\cdot,\cdot\rangle$ on V if and only if it satisfies the *parallelogram identity*

$$\|x+y\|^2 + \|x-y\|^2 = 2\|x\|^2 + 2\|y\|^2$$

for all $x, y \in V$.

(b) Let V be a finite-dimensional $\mathbb{R}$ vector space. Consider the following map:

$$\begin{array}{cccc} ||\cdot||_1 : & V & \to & \mathbb{R}_{\geqslant 0} \\ & v = (v_1, v_2, \ldots, v_n) & \mapsto & \sum_{i=1}^n |v_i| \end{array}$$

Check that $||\cdot||_1$ defines a norm on V and prove that it does not come from a scalar product.

6. Let $K = \mathbb{R}$, $V = M_{n\times n}(K)$, and consider the map

$$\begin{array}{ccc} V \times V & \to & K \\ (A, B) & \mapsto & \mathrm{Tr}(A^T B). \end{array}$$

Show that it defines an inner product on V and find an orthonormal basis with respect to this inner product. The induced norm is called the *Hilbert-Schmidt norm.* Give a formula form the norm of a matrix $A \in V$ in terms of its entries.