D-MATH | Lineare Algebra II | FS 2023

Prof. M. Einsiedler
Prof. P. Biran

Serie 20

Gram-Schmidt, Orthogonality

1. Let V be a finite dimensional euclidean vector space and $S \subset V$ an orthnormal set. Show that we can extend S to an orthomormal basis of V.

2. Let $A \in M_{n\times n}(\mathbb{R})$. Show:

   (a) The matrix $A^T A$ is symmetric.

   (b) The matrix $A^T A$ is positive definite if and only if A is invertible.

   (c) We have $\operatorname{Rang}(A^T A) = \operatorname{Rang}(A)$.

   (d) Assume that A is symmetric and that it admits $(\lambda_1, v_1), (\lambda_2, v_2)$ with $\lambda_1, \lambda_2 \neq 0$ as pairs of eigenvalue-eigenvector. Show that if $\lambda_1 \neq \lambda_2$ then $v_1 \perp v_2$.

3. Compute a decomposition $A = QR$ of the matrix

$$A = \begin{pmatrix} -1 & 1 & -1 \\ 2 & 0 & 1 \\ -2 & 1 & 0 \end{pmatrix}$$

   into an orthogonal matrix Q and an upper triangular matrix R. Use this decomposition to solve the linear system $Ax = b$ for $b = (0, 3, -3)^T$.

4. Let U be the subspace of $\mathbb{R}^3$ with the standard inner product spanned by the two vectors $v_1 = (1, 1, 1)^T$ and $v_2 = (0, 2, 1)^T$.

   (a) Determine an orthonormal basis of U and an orthonormal basis of $U^\perp$.

   (b) Compute the representation matrices of the orthogonal projections $\mathbb{R}^3 \to U$ and $\mathbb{R}^3 \to U^\perp$ with respect to the standard basis of $\mathbb{R}^3$ and the bases from (a).

5. For each of the following vector spaces V endowed with the inner product $\langle \cdot, \cdot \rangle$, find $U^{\perp}$ for the given subset U:

   (a) First consider

$$V = \left\{ (a_0, a_1, a_2, \dots) \mid \sum_{n=0}^{\infty} |a_n|^2 < \infty \right\}, \quad \langle (a_n)_{n=0}^{\infty}, (b_n)_{n=0}^{\infty} \rangle = \sum_{n=0}^{\infty} a_n \cdot b_n,$$

$$U = \{ (a_n)_{n=0}^{\infty} \in V \mid \exists N \geqslant 0 \text{ s.t. } \forall m \geqslant N : a_m = 0 \}$$

   (b) Secondly, we set

$$V = C([0,1]), \quad \langle f, g \rangle = \int_0^1 f(x) \cdot g(x) dx,$$

$$U = \left\{ f \in V \mid \int_0^{1/2} f(x) dx = 0 \right\}.$$

6. For a finite-dimensional Euclidean vector space $(V, \langle\ ,\ \rangle)$ consider the isomorphism

$$\delta \colon V \to V^* := \mathrm{Hom}_{\mathbb{R}}(V, \mathbb{R}),\ v \mapsto \delta(v) := \langle v, \cdot \rangle.$$

   (a) Show that there exists exactly one inner product $\langle\ ,\ \rangle^*$ on V^* such that δ is an isometry.

   (b) Let B be an ordered basis of V, and let B^* be the corresponding dual basis of V^*. Give the representation matrix of $\langle\ ,\ \rangle^*$ with respect to B^* in terms of the representation matrix of $\langle\ ,\ \rangle$ with respect to B.

Single Choice. In each exercise, exactly one answer is correct.

1. Consider the vector space $\mathbb{R}^2$ endowed with the standard scalar product $\langle \cdot, \cdot \rangle$ and the corresponding norm $|| \ ||$. For which vectors $v, w \in \mathbb{R}^2$ do we have $||v + w|| = ||v|| + ||w||$?
   (a) $v = \binom{1}{1}, \ w = \binom{1}{0}$
   (b) $v = \binom{2}{1}, \ w = \binom{1}{2}$
   (c) $v = \binom{2}{3}, \ w = \binom{3}{1}$
   (d) $v = \binom{1}{2}, \ w = \binom{2}{4}$

2. Let V be a euclidean vector space and let $v_1, v_2, v_3 \in V$. Which assertion does in general not hold?
   (a) From $v_1 \perp v_2$ and $v_2 \perp v_3$ it follows that $v_1 \perp v_3$.
   (b) From $v_1 \perp v_2$ and $v_1 \perp v_3$ it follows that $v_1 \perp (v_2 + v_3)$.
   (c) From $v_1 \perp v_2$ it follows that $v_1 \perp -v_2$.
   (d) From $v_1 \perp (v_2 + v_3)$ and $v_1 \perp v_2$ it follows that $v_1 \perp v_3$.

3. Let V be a euclidean vector space and let $S, T \subset V$ be two subsets. Which of the following properties is not equivalent to the other three?
   (a) $S \subset T^{\perp}$
   (b) $T \subset S^{\perp}$
   (c) $S \perp T$
   (d) $\langle S \rangle \cap \langle T \rangle = \{0\}$

4. Let S be a subset of a finite dimensional euclidean vector space V. Which asstertion does not hold in general?
   (a) $(S^{\perp})^{\perp} = \langle S \rangle$.
   (b) S is the orthogonal complement of a subspace of V.
   (c) $S^{\perp}$ is a subspace of V.
   (d) $V = S^{\perp} \oplus (S^{\perp})^{\perp}$.

Multiple Choice Questions

1. Which of the following matrices are hermitian?
   (a) $\begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}$
   (b) $\frac{1}{i}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$
   (c) $\frac{1}{\sqrt{5}}\begin{pmatrix} i & -2 \\ 2 & i \end{pmatrix}$
   (d) $\begin{pmatrix} 0 & i \\ -i & 0 \end{pmatrix}$
2. Which of the following matrices are unitary?
   (a) $\begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}$
   (b) $\frac{1}{i}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$
   (c) $\frac{1}{\sqrt{5}}\begin{pmatrix} i & -2 \\ 2 & i \end{pmatrix}$
   (d) $\begin{pmatrix} 0 & i \\ -i & 0 \end{pmatrix}$
3. Let U, V be unitary $n \times n$ matrices, and let $\lambda = e^{2\pi i\theta}$, $\theta \in \mathbb{R}$. In general, which of the following statements hold?
   (a) $U + V$ is unitary.
   (b) λU is unitary.
   (c) U^{-1} is unitary.
   (d) UV is unitary.