Serie 21

Gram-Schmidt, Orthogonality

1. Let $K = \mathbb{R}, \mathbb{C}$, $A, B \in M_{n\times m}(K)$, $C \in M_{m\times p}(K)$. Prove the following properties of the adjoint matrix:

   (a) $\overline{A+B}^T = \overline{A}^T + \overline{B}^T$;

   (b) For all $\lambda \in K$, $\overline{(\lambda A)}^T = \overline{\lambda}\,\overline{A}^T$;

   (c) $\overline{(\overline{A}^T)}^T = A$;

   (d) $\overline{I_n}^T = I_n$;

   (e) $\overline{(A \cdot C)}^T = \overline{C}^T \cdot \overline{A}^T$.

2. Let $K = \mathbb{R}$. On $K[x]_2$, consider the inner product given by

$$\langle p, q\rangle = \int_0^1 p(x)q(x)dx.$$

   (a) Apply the Gram-Schmidt procedure to the basis $1, x, x^2$ to produce an orthonormal basis of $K[x]_2$.

   (b) Find an orthonormal basis of $K[x]_2$ such that the differential operator $p \mapsto p'$ on $K[x]_2$ has an upper triangular matrix with respect to this basis.

3. **Minimizing the distance to a subset.** Let K be a field and V be a K-vector space. Suppose that U is a finite-dimensional subspace of V and denote $P_U : V \to U$ the orthogonal projection onto U. Let $v \in V$ and $u \in U$. Show that

$$||v - P_U(v)|| \leqslant ||v - u||\,.$$

   Additionally, prove that the inequality above is an equality if and only if $u = P_U(v)$.

4. Find a polynomial p with real coefficients and degree at most 5 that approximates $\sin(x)$ as well as possible on the interval $[-\pi, \pi]$, in the sense that

$$\int_{-\pi}^{\pi} |\sin(x) - p(x)|^2\, dx$$

   is as small as possible.

Hint. Reformulate the problem in order to use exercise 3.

5. Let $V = C([-1,1],\mathbb{R})$ denote the space of continuous real-valued functions on the interval $[-1,1]$ with inner product

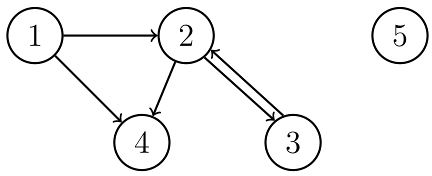
$$\langle f, g\rangle = \int_{-1}^{1} f(x)g(x)dx,$$

for $f, g \in V$. Let $\varphi : V \to \mathbb{R}$ be the linear functional defined by $\varphi(f) = f(0)$. Show that there does not exist $g \in V$ such that

$$\forall f \in V : \varphi(f) = \langle f, g\rangle.$$

6. Let $G = (V, E)$ be a finite directed graph. More precisely, let V be a finite set, and let $E \subseteq \{(v_{\text{init}}, v_{\text{term}}) \mid v_{\text{init}}, v_{\text{term}} \in V \wedge v_{\text{init}} \neq v_{\text{term}}\} \subseteq V \times V$. We think of V as the set of vertices of the graph, and of the pair $(v_{\text{init}}, v_{\text{term}}) \in E$ as the directed edge connecting $v_{\text{init}} \in V$ to $v_{\text{term}} \in V$ (this can be represented by drawing an arrow pointing towards v_{term} on the said edge).

   Example of a directed graph.



   We also define the vector spaces $\mathbb{R}^V = \{f : V \to \mathbb{R}\}$ and $\mathbb{R}^E = \{\varphi : E \to \mathbb{R}\}$, which we equip with the inner products

$$\langle f_1, f_2 \rangle_V = \sum_{v \in V} f_1(v) f_2(v), \quad f_1, f_2 \in \mathbb{R}^V$$
$$\langle \varphi_1, \varphi_2 \rangle_E = \sum_{e \in E} \varphi_1(e) \varphi_2(e), \quad \varphi_1, \varphi_2 \in \mathbb{R}^E.$$

   Also define $T : \mathbb{R}^V \to \mathbb{R}^E$ as the "combinatorial derivative": for $f \in \mathbb{R}^V$ and $e = (v_{\text{init}}, v_{\text{term}}) \in E$, let

$$T(f)(e) = f(v_{\text{term}}) - f(v_{\text{init}}).$$

   Also define $S : \mathbb{R}^E \to \mathbb{R}^V$ by

$$S(\varphi)(v) = \sum_{\substack{v_{\text{init}} \in V \\ (v_{\text{init}}, v) \in E}} \varphi((v_{\text{init}}, v)) - \sum_{\substack{v_{\text{term}} \in V \\ (v, v_{\text{term}}) \in E}} \varphi((v, v_{\text{term}})).$$

   (a) Show that $T^* = S$ and calculate $T^* \circ T = S \circ T$, which is also called the combinatorial Laplacian of G.

   (b) Now simplify the setup by assuming that the graph is undirected, i.e.

$$(v_{\text{init}}, v_{\text{term}}) \in E \Leftrightarrow (v_{\text{term}}, v_{\text{init}}) \in E,$$

   and d-regular (for any $v \in V$ there are exactly d vertices $v_{\text{term}} \in V$ with $(v, v_{\text{term}}) \in E$). Show that $T^* \circ T$ admits 0 as an eigenvalue. Explain why the geometric multiplicity of 0 is related to the connectivity of G.

Single Choice. In each exercise, exactly one answer is correct.

1. For which $x \in \mathbb{C}$ is the matrix $A := \begin{pmatrix} x & -x \\ x & x \end{pmatrix}$ unitary?

   (a) For all $x \in \mathbb{C}$ with $|x|^2 = \frac{1}{2}$.

   (b) Exactly for $x = \frac{1}{\sqrt{2}}$.

   (c) For all $x \in \mathbb{C}$ with $x = -\overline{x}$.

   (d) For $x = 0$.

2. Which set is a subspace of the $\mathbb{C}$-vector space $M_{n \times n}(\mathbb{C})$?

   (a) The set of unitary $n \times n$ matrices.

   (b) The set of self-adjoint $n \times n$ matrices.

   (c) The set of symmetric $n \times n$ matrices.

   (d) The set of normal $n \times n$ matrices.

Multiple Choice Fragen

1. Let A be a Hermitian matrix. Which statements are correct?
   (a) $\mathrm{Tr}(A) \in \mathbb{R}$.
   (b) $\det(A) \in \mathbb{R}$.