Serie 22

Self-adjoint operators, Spectral theory

1. Let K be a field and let $(V, \langle\cdot,\cdot\rangle)$ be an inner product finite-dimensional K-vector space. Suppose that $T \in \mathrm{End}(V)$ and that U is a subspace of V. Prove that U is invariant under T if and only if $U^\perp$ is invariant under T^*.

2. Let f, g_1, and g_2 be endomorphisms of a finite-dimensional Euclidean vector space such that
$$f^* \circ f \circ g_1 = f^* \circ f \circ g_2\,.$$
Prove that $f \circ g_1 = f \circ g_2$.

3. Let V be a finite-dimensional unitary vector space and let $T \in \mathrm{End}(V)$ be a normal operator. For a subspace $W \subseteq V$, we denote P_W the orthogonal projection onto W.

   (a) Show the following:

   Theorem. *There exist finitely many complex numbers $\lambda_1, \dots, \lambda_k \in \mathbb{C}$, and mutually orthogonal subspaces of V denoted $W_1, \dots, W_k$ such that*
   $$T = \lambda_1 P_{W_1} + \cdots + \lambda_k P_{W_k}.$$

   (b) Show that for any subspace U of V, P_U is self-adjoint.

4. Make $\mathbb{R}[x]_2$ into an inner product space by defining
$$\langle p, q\rangle = \int_0^1 p(x)q(x)dx$$
Define $T \in \mathrm{End}(\mathbb{R}[x]_2)$ by $T(a_0 + a_1x + a_2x^2) = a_1x$.

   (a) Show that T is not self-adjoint.

   (b) The matrix of T with respect to the basis $(1, x, x^2)$ is
   $$\begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$
   This matrix equals its conjugate transpose, even though T is not self-adjoint. Explain why this is not a contradiction.

5. Let $f : V \to W$ be a homomorphism of euclidian vector spaces.

   (a) Assume that $\dim(V) < \infty$. Show that the adjoint of f exists in the following sense: show that there exists a unique map $f' : W \to V$ such that

   $$\text{for all } v \in V, \text{ for all } w \in W : \langle f(v), w \rangle = \langle v, f'(w) \rangle.$$

   (b) Does the statement still hold if instead of assuming $\dim(V) < \infty$ we assume that $\dim(W) < \infty$?

6. Consider the vector space V consisting of all infinitely differentiable periodic functions from $\mathbb{R}$ to $\mathbb{R}$ with period 2π, equipped with the inner product

   $$\langle f, g \rangle = \frac{1}{2\pi} \int_0^{2\pi} f(x)g(x)dx.$$

   Let $D : V \to V$ be the linear transformation defined by $D(f) = \frac{df}{dx}$.

   (a) Is D self-adjoint? Determine its adjoint if it exists.

   (b) Is $\Delta := -D \circ D$ self-adjoint?

   (c) Let $U \subset V$ be the linear span of the functions

   $$\{x \mapsto \cos(nx) \mid n \in \mathbb{Z}\} \cup \{x \mapsto \sin(nx) \mid n \in \mathbb{Z}\},$$

   with the induced inner product from V. Find an orthonormal basis of U consisting of eigenvectors of $\Delta|_U$ and the multiplicities of all eigenvalues.

Multiple Choice Fragen

1. Let A and B be complex self-adjoint $n \times n$ matrices, and let $\lambda \in \mathbb{C}$. Which of the following statements hold?
   (a) $A + B$ is self-adjoint.
   (b) λA is self-adjoint.
   (c) λA is normal.
2. Let A, B be complex self-adjoint $n \times n$ matrices and let $\lambda \in \mathbb{C}$. Which of the following statements hold?
   (a) AB is self-adjoint.
   (b) $AB + BA$ is self-adjoint.
   (c) $AB - BA$ is normal.
   (d) ABA is self-adjoint.
3. Let A be a normal matrix and $p \in \mathbb{C}[t]$ be a polynomial. Which of the following statements hold?
   (a) $p(A)^* = p(A^*)$.
   (b) $A^i(A^*)^j = (A^*)^j A^i$.
   (c) $p(A)$ is normal.
   (d) Every eigenvalue λ of A is also an eigenvalue of $p(A)$.
   (e) Every eigenvector v of A is also an eigenvector of $p(A)$.