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Serie 23

Positive-definiteness, isometries

1. Let K be a field in which $2 \neq 0$, V a K-vector space, and let B be a symmetric bilinear 9form on V. We define $q_B(v) = B(v,v)$ for every $v \in V$ to be the quadratic form associated to B. Show that

$$B(v,w) = \frac{1}{2}(q_B(v+w) - q_B(v) - q_B(w)).$$

2. Consider the real matrix

$$A := \frac{1}{3}\begin{pmatrix} 2 & -2 & 1 \\ -1 & -2 & -2 \\ 2 & 1 & -2 \end{pmatrix}.$$

   (a) Show that A is orthogonal and $\det A = 1$.

   (b) Determine the rotational axis and the angle of $T_A : \mathbb{R}^2 \to \mathbb{R}^2$, $v \mapsto Av$.

3. Which of the following three real symmetrix matrices are positive definite?

$$A := \begin{pmatrix} 3 & 3 & 2 & 3 \\ 3 & 1 & 1 & 2 \\ 2 & 1 & 2 & 1 \\ 3 & 2 & 1 & 3 \end{pmatrix}, \quad B := \begin{pmatrix} 6 & 3 & 4 \\ 3 & 7 & 3 \\ 4 & 3 & 8 \end{pmatrix}, \quad C := \begin{pmatrix} 3 & 0 & -1 & 0 \\ 0 & 6 & 1 & 1 \\ -1 & 1 & 8 & 2 \\ 0 & 1 & 2 & 5 \end{pmatrix}.$$

   Hinweis: Verwende das Hauptminorenkriterium.

4. Let $A \in M_{n\times n}(\mathbb{R})$ be a symmetric matrix. Show that the following statements are equivalent:

   (A) A is positive definite, i.e. $v^T Av > 0$ for all $v \neq 0$;

   (B) All eigenvalues of A are positive;

   (C) There exists an invertible symmetric matrix $S \in M_{n\times n}(\mathbb{R})$ such that $S^2 = A$.

5. Show: For every orthogonal endomorphism f of an n-dimensional Euclidean vectorspace V, we have

$$|\operatorname{Tr}(f)| \leqslant n.$$

   For which f do we have equality?

6. Consider two 2-dimensional subspaces $E_1, E_2 \subset \mathbb{R}^3$. Describe the set of elements $T \in \mathrm{SO}_3(\mathbb{R})$ such that

$$TE_1 = E_2,$$

in terms of orthogonal bases of E_1 and E_2.

Hint: Start by assuming that $E_1 = E_2 = \mathrm{Sp}(e_1, e_2)$.