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Serie 24

Bilinear forms, Singular values decomposition, Jordan Normal form

1. (a) Determine a singular value decomposition $A = QDR$ of the real matrix

$$A := \begin{pmatrix} 1 & -1 \\ 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

   (b) Determine a singular value decomposition of A^T.

2. Consider the real matrix

$$A = \begin{pmatrix} 14 & -13 & 8 \\ -13 & 14 & 8 \\ 8 & 8 & -7 \end{pmatrix} \in M_{3\times 3}(\mathbb{R}).$$

   Find a matrix $P \in \mathrm{O}_3(\mathbb{R})$ such that $P^{-1}AP$ is diagonal.

3. Let V be an n-dimensional vector space over a field K. Show the following statements using the lemma about generalized eigenspaces seen in the lectures, but without using the Jordan Normal Form theorem:

   (a) Suppose that $N \in \mathrm{End}(V)$ is nilpotent. Then 0 is an eigenvalue of N and it is the only one.

   (b) If $N \in \mathrm{End}(V)$ is nilpotent, then $N^n = O_{n\times n}$. In other words, the nilpotency index of N is smaller or equal to $\dim(V)$.

   (c) Suppose that $N \in \mathrm{End}(V)$ is nilpotent and assume that $p_N(x)$ splits as a product of linear factors in $K[x]$. Then $p_N(x) = (-x)^n$.

   (d) Let $T \in \mathrm{End}(V)$ and assume that $p_T(x)$ splits into linear factors in $K[x]$. Let $\eta \in K$ and define $S = T - \eta\,\mathrm{Id}_V$. Then $p_S(x)$ also splits into a product of linear factors over $K[x]$. In fact, $p_S(x) = p_T(x + \eta)$.

   (e) Let $T \in \mathrm{End}(V)$ and assume that $\lambda \in K$ is the only eigenvalue of T and that $p_T(x)$ splits as a product of linear factors in $K[x]$. Define $N = T - \lambda\,\mathrm{Id}_V$. Then $p_N(x) = (-x)^n$, and $N^n = O_{n\times n}$.

4. Determine the Jordan normal form of the following matrix over $\mathbb{R}$ and over $\mathbb{F}_3$:

$$A := \begin{pmatrix} 2 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 1 & 2 \end{pmatrix}$$

5. Determine the Jordan normal form and the corresponding base change matrices of the real matrix

$$A := \begin{pmatrix} 2 & 2 & 2 & 2 \\ 0 & 3 & 0 & 2 \\ 0 & 0 & 3 & 2 \\ 0 & 0 & 0 & 3 \end{pmatrix}.$$

6. *Example regarding special relativity.* Define the symmetric blinear forms$: \mathbb{R}^4 \to \mathbb{R}^4$ for all $v = (x, y, z, t)^T$ and $v' = (x', y', z', t')^T$ in $\mathbb{R}^4$ by

$$s(v, v') := xx' + yy' + zz' - ctt',$$

where $c > 0$ is a fixed parameter. The space $M := (\mathbb{R}^4, s)$ is called *Minkowski space* (sometimes Minkowski spacetime) and the parameter c is called *light speed.* We use the normalization $c = 1$.

A linear map $F : M \to M$ is called *isometry* or *Lorentz transformation*, if

$$\forall v, w \in \mathbb{R}^4 \colon \; s\big(F(v), F(w)\big) = s(v, w)\,.$$

(a) Show that every isometry is bijective.

(b) Show that the following endomorphisms are isometries of M:

i. Left multiplication with $\left(\begin{array}{c|c} T & 0 \\ \hline 0 & \pm 1 \end{array}\right)$ für jedes $T \in O(3)$.

ii. A *Lorentz boost* in x-direction with speed $v < c = 1$, given by left multiplication with the matrix

$$B \;:=\; \begin{pmatrix} \gamma & & & -v\gamma \\ & 1 & & \\ & & 1 & \\ -v\gamma & & & \gamma, \end{pmatrix}$$

for $\gamma := 1/\sqrt{1 - v^2}$.

(c) The subset $\{x \in M \mid s(x, x) = 0\}$ is called *light cone in M*. Prove the „relativistic football theorem“: Every linear isometry φ with $\det(\varphi) = 1$ has an eigenvector in the light cone.

Remark. For $c \to \infty$ the light cone approaches the subspace $\{t = 0\}$ and the statement reduces to the classical case.

Single Choice. In each exercise, exactly one answer is correct.

1. Let f be an endomorphism of a finite-dimensional vector space V and let λ be an eigenvalue of f. Which statement is generally false?

   (a) Every eigenvector of f corresponding to the eigenvalue λ lies in the eigenspace $\widetilde{\mathrm{Eig}}_f(\lambda)$.

   (b) Every vector in $\widetilde{\mathrm{Eig}}_f(\lambda)$ is an eigenvector of f corresponding to the eigenvalue λ.

   (c) The generalized eigenspace $\widetilde{\mathrm{Eig}}_f(\lambda)$ is not the zero space.

   (d) For every eigenvalue μ of f with $\mu \neq \lambda$, we have $\widetilde{\mathrm{Eig}}_f(\mu) \cap \widetilde{\mathrm{Eig}}_f(\lambda) = \langle 0 \rangle$.

2. For every endomorphism f of an n-dimensional vector space V whose characteristic polynomial factors into linear factors, and every eigenvalue λ of f we have:

   (a) $\widetilde{\mathrm{Eig}}_f(\lambda) = \mathrm{Kern}(f - \lambda\,\mathrm{id}_V)$.

   (b) $\dim(\widetilde{\mathrm{Eig}}_f(\lambda)) = 1$.

   (c) $\dim(\widetilde{\mathrm{Eig}}_f(\lambda)) = n$.

   (d) $\widetilde{\mathrm{Eig}}_f(\lambda) = \mathrm{Kern}((f - \lambda\,\mathrm{id}_V)^n)$.

3. The generalized eigenspace of the real matrix $A := \begin{pmatrix} 2 & 3 & -1 & 5 \\ 0 & 2 & -3 & 1 \\ 0 & 0 & 2 & -2 \\ 0 & 0 & 0 & 3 \end{pmatrix}$ with respect to $X - 2$ is

   (a) One-dimensional

   (b) Two-dimensional

   (c) Three-dimensional

   (d) Four-dimensional

4. Let A be a 3×3 matrix with $A \neq 0$ and $A^2 = 0$. Then, the Jordan normal form of A has

   (a) 1 Jordan block.

   (b) 2 Jordan blocks.

   (c) 3 Jordan blocks.

   (d) It depends on the exact matrix A.

Multiple Choice Fragen

1. Which of the following statements is **true**: For arbitrary integers $n > m \geqslant 1$, there exists a square matrix with...

   (a) characteristic polynomial $X^m + X^n$.

   (b) minimal polynomial X^m and characteristic polynomial X^n.

   (c) minimal polynomial $X^m \cdot (X^n - 1)$.