Serie 25

Jordan Normal form, Multilinear algebra

1. Prove the following propositions:

   (a) For all K-vector spaces $V_1, \ldots, V_r$ and W, we have that $\mathrm{Mult}_K(V_1, \ldots, V_r; W)$ is a subspace of the vector space of all maps $V_1 \times \cdots \times V_r \to W$.

   (b) Consider linear maps of K-vector spaces $f_i \colon V_i' \to V_i$ for $1 \leqslant i \leqslant r$ as well as $g \colon W \to W'$. Then we get a linear map

$$\begin{aligned} \mathrm{Mult}_K(V_1, \ldots, V_r; W) &\to \mathrm{Mult}_K(V_1', \ldots, V_r'; W'), \\ \varphi &\mapsto g \circ \varphi \circ (f_1 \times \cdots \times f_r). \end{aligned}$$

2. Let K be a field. Consider the space $K[x]_n$ of polynomials over K of degree at most n.

   (a) Find a Jordan normal form for the endomorphism

$$D : \begin{array}{ccc} K[x]_n & \to & K[x]_n \\ p(x) & \mapsto & p'(x) \end{array}$$

   (b) Find a Jordan normal form for the endomorphism

$$D_2 : \begin{array}{ccc} K[x]_n & \to & K[x]_n \\ p(x) & \mapsto & p''(x) \end{array}$$

3. Determine a Jordan normal form over $\mathbb{C}$ of the matrix

$$A := \begin{pmatrix} -1 & -1 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \\ -2 & 0 & 1 & 1 \end{pmatrix}$$

4. Often the Jordan normal form is motivated by the desire to have a matrix with as much zeros as possible. Is the number of zeros actually maximized by the Jordan normal form? Stated differently: Does there exist a square matrix A over a field which has more zeros than its Jordan normal form J?

5. Let B be a complex 5×5-matrix with minimaly polynomial $(X-3)(X+5)^2$ and characteristic polynomial $(X-3)^2(X+5)^3$. Determine all possible Jordan normal fomrs of B.

6. Let A be a real square matrix. We define the exponential of such a matrix as

$$\exp(A) = \sum_{k=0}^{\infty} \frac{A^k}{k!}$$

when it converges.

   (a) For $\lambda \in \mathbb{R}$ and $n \geqslant 1$, compute $\exp(J_{\lambda,n})$.

   (b) Determine the solution of the system of differential equations

$$\begin{aligned} x'(t) &= -x(t) + 9y(t) + 9z(t) \\ y'(t) &= 3x(t) - 6y(t) - 8z(t) \\ z'(t) &= -4x(t) + 11y(t) + 13z(t) \end{aligned}$$

   with the initial conditions $x(0) = y(0) = z(0) = 1$.

   Hint: Use the Jordan normal form. If you need more hints, have a look at Chapter 9.5 from Menny Akka's notes.

   (c) Determine the general real solution of the differential equation

$$f^{(3)}(t) - f^{(2)}(t) + f'(t) - f(t) = 0.$$

   Hint: Write the equation as a system of linear differential equations of the first order and use the Jordan normal form.