D-MATH
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Lineare Algebra II

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Serie 26

Multilinear algebra, Tensor product

1. Simplify the following expression in $\mathbb{R}^2 \otimes \mathbb{R}^3$.

$$-\begin{pmatrix}1\\2\end{pmatrix} \otimes \begin{pmatrix}3\\-1\\2\end{pmatrix} + \begin{pmatrix}3\\2\end{pmatrix} \otimes \begin{pmatrix}1\\0\\1\end{pmatrix} + 2\begin{pmatrix}1\\1\end{pmatrix} \otimes \begin{pmatrix}2\\-1\\1\end{pmatrix} - \begin{pmatrix}1\\0\end{pmatrix} \otimes \begin{pmatrix}5\\0\\1\end{pmatrix}$$

2. Let K be a field. Consider the bilinear map

$$\begin{array}{cccc} \psi: & K^m_{\text{cols}} \times K^n_{\text{cols}} & \to & M_{m\times n}(K) \\ & (u,v) & \mapsto & u \cdot v^T \end{array}$$

   (a) Prove that the image of ψ is the set of matrices of rank lower or equal to 1.

   (b) Is $\mathrm{Im}(\psi)$ a linear subspace?

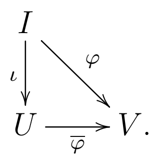
   (c) Describe $\mathrm{Sp}(\mathrm{Im}(\psi))$.

3. Let V, W be finite-dimensional vector spaces over a field K. Show that

$$V^* \otimes W \cong \mathrm{Hom}(V, W).$$

4. Consider a set I. For a pair (U, ι) consisting of a K-vectorspace U and a map $\iota\colon I \to U$ consider the following *universal property*:

   For every K-vectorspace V and for every map $\varphi\colon I \to V$ there exists exactly one linear map $\overline{\varphi}\colon U \to V$, such that the following diagram commutes:



   (a) Show that for two pairs (U, ι) and (U', ι') satisfying the universal property, there exists a unique isomorphism $\psi\colon U \xrightarrow{\sim} U'$ with $\psi \circ \iota = \iota'$.

   (b) Show that the universal property is satisfied for the K-vectorspace

$$K^{(I)} := \left\{ (x_i)_{i\in I} \in K^I \;\middle|\; x_i \neq 0 \text{ for at most finitely many } i \right\}$$

   with the map

$$\iota_I : I \to K^{(I)}, \quad i \mapsto (\delta_{ij})_{j\in I}\,.$$

5. Prove the following statements concerning vector spaces U, V, V_1, V_2 over a field K:

   (a) There is a unique linear map $\kappa : U \otimes V \to V \otimes U$ that satisfies

   $$\kappa(u \otimes v) = v \otimes u.$$

   This map is an isomorphism.

   (b) There is a canonical isomorphism between $U \otimes K$ and U.

   (c) There is a canonical isomorphism

   $$U \otimes (V_1 \oplus V_2) \to (U \otimes V_1) \oplus (U \otimes V_2).$$

   Remark. Do not define the required homomorphisms with respect to bases.

6. Let V be a vector space of dimension $n < \infty$, and let f be an endomorphism of V with characteristic polynomial $\mathrm{char}_f(X) = \sum_{i=0}^{n} a_i X^i$. For all $r > 0$ consider the induced map

   $$\mathrm{Alt}^r(f)\colon\ \mathrm{Alt}^r_K(V, K) \to \mathrm{Alt}^r_K(V, K),\ \varphi \mapsto \varphi \circ (f \times \ldots \times f).$$

   Show: For all $r = 1, \ldots, n$, we have

   $$a_{n-r} = (-1)^{n+r}\,\mathrm{Tr}\,\mathrm{Alt}^r(f).$$