Lineare Algebra I/II - Mock exam

1. *(5 points)*

   (a) (2 points) Give the definition of the symmetric group on n-elements S_n. For a permutation $\sigma \in S_n$ give the defintion of $\text{sign}(\sigma)$. You do not have to show that $\text{sign}(\sigma)$ is well defined.

   (b) (3 points) Let $\sigma \in S_n$ and let $A \in M_{n\times n}(K)$ be the matrix obtained from the identity matrix I_n by permuting its rows using the permutation σ. Prove that $\det(A) = \text{sign}(\sigma)$.

   Solution:

   (a) The definition of S_n is

   $$S_n = \{\sigma : \{1, \dots, n\} \to \{1, \dots, n\} \mid \sigma \text{ is bijective}\}.$$

   Let $\sigma \in S_n$. We have seen in the lectures that σ can always be written as a composition of finitely many transpositions, i.e. $\sigma = \tau_1 \cdots \tau_m$, where $\tau_1, \dots, \tau_m$ are transpositions. Then $\text{sign}(\sigma)$ is defined to be $(-1)^m$.

   (b) We prove this by induction on the number of transpositions m used to decompose σ. First assume that $m = 1$. Then σ exchanges two rows of the identity. Hence, by properties of the determinant, we have $\det(A) = -\det(I_n) = \text{sign}(\sigma)$.

   Now assume that $m > 1$ and that we have shown the claim for all permutations that can be decomposed as a product of at most $m - 1$ transpositions. Assume that $\sigma = \tau_1 \cdots \tau_m$, where the τ_i's are transpositions. Let us denote B the matrix obtained by permuting the rows of the identity matrix using $\tau_2 \cdots \tau_m$. It follows from the induction hypothesis that $\det(B) = \text{sign}(\tau_2 \cdots \tau_m) = (-1)^{m-1}$. Now, note that A is obtained by permuting two rows of B using τ_1. Hence,

   $$\det(A) = -\det(B) = (-1)^m = \text{sign}(\sigma).$$

2. *(5 points)* Let V be a finite dimensional vector space over the field $\mathbb{K}$. Let $T \in End(V)$ and let $\lambda_1, \lambda_2 \in \mathbb{K}$ be two distinct eigenvalues of T. Denote by $Eig_T(\lambda_i)$ the eigenspace of λ_i and by $\widetilde{Eig}_T(\lambda_i)$ the generalized eigenspace of λ_i, $i = 1, 2$.

   (a) (2 points) Prove that $Eig_T(\lambda_1) \cap Eig_T(\lambda_2) = \{0\}$.

   (b) (3 points) Assume that the characteristic polynomial $p_T(x)$ of T splits as a product of linear factors in $\mathbb{K}[x]$. Prove that $\widetilde{Eig}_T(\lambda_1) \cap \widetilde{Eig}_T(\lambda_2) = \{0\}$.

   Solution:

(a) Assume by contradiction that there exists a non-zero vector $v \in Eig_T(\lambda_1) \cap Eig_T(\lambda_2)$. Then,

$$\lambda_1 v = Tv = \lambda_2 v \implies (\lambda_1 - \lambda_2)v = 0 \implies \lambda_1 = \lambda_2,$$

which is a contradiction.

(b) Let $\lambda \in \mathbb{K}$ be an eigenvalue of T. We first claim that every eigenvector v of T which belongs to $\widetilde{Eig}_T(\lambda)$ must have eigenvalue λ. Indeed, if μ is the eigenvalue of v, then $Tv = \mu v$, hence $T^k v = \mu^k v$ for all $k \geqslant 0$. It follows that

$$(\lambda id_V - T)^k v = (\lambda - \mu)^k v, \text{ for all } k \geqslant 0.$$

In particular, for $n := \dim(V)$ we have $(\lambda id_V - T)^n v = (\lambda - \mu)^n v$. On the other hand, by a known result

$$\widetilde{Eig}_T(\lambda) = \mathrm{Ker}((\lambda id_V - T)^n),$$

hence $(\lambda - \mu)^n v = 0$. As $v \neq 0$, it follows that $(\lambda - \mu)^n = 0$, hence $\lambda = \mu$. This proves the claim.

Let λ be as above. Recall that $\widetilde{Eig}_T(\lambda)$ is a T-invariant subspace. Consider $T_\lambda := T|_{\widetilde{Eig}_T(\lambda)}$. By a known result, $p_{T_\lambda}(x)$ divides $p_T(x)$. Since $p_T(x)$ splits as a product of linear factors in $\mathbb{K}[x]$, so does $p_{T_\lambda}(x)$. As λ is the only eigenvalue of T_λ, it follows that $p_{T_\lambda}(x)$ has the form

$$p_{T_\lambda}(x) = (\lambda - x)^{r_\lambda},$$

for some $r_\lambda \geqslant 1$.

We are now ready to prove the statement of the problem. Put $U := \widetilde{Eig}_T(\lambda_1) \cap \widetilde{Eig}_T(\lambda_2)$ and assume by contradiction that $U \neq \{0\}$. Then $\ell := \dim(U) \geqslant 1$. Clearly $U \subseteq V$ is a T-invariant subspace because for $i = 1, 2$, $\widetilde{Eig}_T(\lambda_i)$ are T-invariant. Put $T_U := T|_U \in \mathrm{End}(U)$. Note that the characteristic polynomial $p_{T_U}(x)$ of T_U has degree $\ell \geqslant 1$ hence is not the constant polynomial. By a known result, $p_{T_U}(x)$ divides both $p_{T_{\lambda_1}}(x)$ and $p_{T_{\lambda_2}}(x)$. But this is impossibe since we have seen that

$$p_{T_{\lambda_1}}(x) = (\lambda_1 - x)^{r_{\lambda_1}}, \quad p_{T_{\lambda_2}}(x) = (\lambda_2 - x)^{r_{\lambda_2}},$$

and $\lambda_1 \neq \lambda_2$. This yields a contradiction.

3. *(10 Points)* For each statement, mark with a cross whether it is true (T) or false (F). Correct answers are awarded +1 point, incorrect answers or no answer 0 points.

(a) ■T □F The matrix $B = \begin{pmatrix} 0 & \sqrt{3}i & 0 \\ \sqrt{3}i & 0 & 0 \\ 0 & 0 & -3 \end{pmatrix}$ is diagonalisable over $\mathbb{C}$.

(b) □T ■F Every diagonalisable matrix $A \in M_{n,n}(\mathbb{K})$ consists of n linearly independent column vectors.

(c) ☐T ■F Every matrix $A \in M_{n,n}(\mathbb{R})$, whose eigenvalues are all positive, is symmetric.

(d) ■T ☐F Every matrix $A \in SO(3)$ satisfies $\operatorname{tr}(A) \leqslant 3$.

(e) ■T ☐F Let $A \in M_{4,4}(\mathbb{C})$ be a matrix with characteristic polynomial $p_A(x) = (x+i)^2(x-\sqrt{2})(x+2)$. Then A is diagonalisable if and only if $\dim(\operatorname{Ker}(A + i1_4)) = 2$.

(f) ☐T ■F Let f be a vector space endomorphism. Then, for every eigenvalue of f there exists a unique eigenvector.

(g) ■T ☐F Let V be a finite-dimensional vector space, and let V^* be its dual space. Then $V \cong V^*$.

(h) ☐T ■F Let V be the vector space of continuous functions $f : \mathbb{R} \to \mathbb{R}$ such that $f(x+1) = f(x)$ for all $x \in \mathbb{R}$. The map

$$(f, g) = \int_0^1 f(x)g(x + \frac{1}{2})dx$$

defines a scalar product over V.

(i) ■T ☐F Consider $v_1 \in \mathbb{R}^3$ with $\|v_1\| = 1$. There is exactly one vector $v \in \mathbb{R}^3$, such that $(v, v_1) = 1$ and $\|v\| = 1$.

(j) ☐T ■F Consider two endomorphisms f, g of a finite-dimensional euclidian vector space. It holds that

$$f^*g^* = gf \Leftrightarrow fg = g^*f^*.$$

Counterexample: Consider the matrices

$$A = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad B = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \in \operatorname{End}(\mathbb{R}^2).$$

We have

$$AB = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}.$$

Hence, $B^T A^T = (AB)^T = AB$. However,

$$BA = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix},$$

is not symmetric. Hence $A^T B^T = (BA)^T \neq BA$.

4. *(14 Points)* Write your answer directly on the exam sheet. You do not have to justify your answer.

(a) Compute the determinant of

$$A_\lambda = \begin{pmatrix} 1 & -\lambda & \lambda^2 & -\lambda^3 \\ 1 & 2\lambda & 4\lambda^2 & 8\lambda^3 \\ 1 & \lambda & \lambda^2 & \lambda^3 \\ 1 & -2\lambda & 4\lambda^2 & -8\lambda^3 \end{pmatrix} \quad \text{für } \lambda \in \mathbb{C}. \qquad \det(A_\lambda) = \boxed{}$$

Solution: $72\lambda^6$.

(b) For which $\alpha \in \mathbb{C}$ (give all of them) is the matrix

$$A_\alpha = \begin{pmatrix} 1 & -1 & 0 \\ \alpha & -1 & 0 \\ 2 & 1 & \alpha \end{pmatrix} \in M_{3,3}(\mathbb{C}) \text{ invertible?} \qquad \alpha \in \boxed{}$$

Solution: $\alpha \in \mathbb{C} \setminus \{0, 1\}$.

(c) Let A_α be the same as in (b), and let α be such that A_α is invertible. Compute the inverse of A_α. (Your answer will depend on the variable α.)

$$A_\alpha^{-1} = \boxed{}$$

Solution:

$$A^{-1} = \begin{pmatrix} \frac{1}{1-\alpha} & \frac{1}{\alpha-1} & 0 \\ \frac{\alpha}{1-\alpha} & \frac{1}{\alpha-1} & 0 \\ \frac{\alpha+2}{(\alpha-1)\alpha} & \frac{3}{\alpha-\alpha^2} & \frac{1}{\alpha} \end{pmatrix}$$

(d) Compute the eigenvalues of the matrix

$$\begin{pmatrix} 0 & -4 & 6 \\ -3 & 5 & 0 \\ 2 & -4 & 1 \end{pmatrix} \in M_{3,3}(\mathbb{R}). \qquad \text{Antwort: } \boxed{}$$

Solution: $\lambda_1 = 0$, $\lambda_2 = 3 + 2\sqrt{7}$, $\lambda_3 = 3 - 2\sqrt{7}$.

(e) Let $A = \begin{pmatrix} 1 & 0 & 3 \\ -1 & \alpha & -2 \\ 2 & -\alpha & 2 \end{pmatrix}$ and $b = \begin{pmatrix} -3 \\ 1 \\ 2 \end{pmatrix}$. Find all $x \in \mathbb{R}^3$ such that $Ax = b$, for $\alpha \in \mathbb{R} \setminus \{0\}$.

$x \in$ [ ]

Solution: $x = \begin{pmatrix} 3 \\ 0 \\ -2 \end{pmatrix}$.

(f) Compute the minimal polynomial of the matrix

$$A = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 1 & -1 & 1 \end{pmatrix}.$$

Antwort: [ ]

Solution: The minimal polynomial of A is $(x-1)(x+1)$.

(g) Let $T : \mathbb{R}^2 \to \mathbb{R}[x]$ be the unique linear map such that $T(5,2) = 11 + 22x$ and $T(1,7) = 33 - 11x$. Compute $T(1,4)$.

$T(1,4) =$ [ ]

Solution: $T(1,4) = 19 - 4x$.

5. *(10 Points)* Consider the vector subspaces

$$V_1 = \left\langle \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}, \begin{pmatrix} 3 \\ -4 \\ 5 \end{pmatrix} \right\rangle$$

and

$$V_2 = \left\{ \begin{pmatrix} x \\ y \\ z \end{pmatrix} \in \mathbb{R}^3 \middle| x + y + z = 0 \right\}$$

in $V = \mathbb{R}^3$.

(a) (2 Points) Determine the dimension of V_1, V_2 and $V_1 \cap V_2$.

(b) (3 Points) Find a basis of $V_1 \cap V_2$.

(c) (3 Points) Find a basis of the orthogonal complement of $V_1 \cap V_2$ with respect to the standard scalar product over $V = \mathbb{R}^3$.

(d) (2 Points) Find a linear map $f : \mathbb{R}^3 \to \mathbb{R}^3$ with $\mathrm{Ker}(f) = V_1 \cap V_2$.

Solution:

(a) We have $\dim(V_1) = 2$ since the 2 vectors generating V_1 are linearly independent. We observe that $\dim(V_2) \geqslant 2$ since $(1, -1, 0)^T \in V_2$ and $(1, 0, -1)^T \in V_2$ are linearly independent. Moreover, $V_2 \subsetneq \mathbb{R}^3$ because for example $(1, 0, 0)^T \notin V_2$. So $2 \leqslant V_2 < 3$, hence $\dim(V_2) = 2$. Finally assume that a general vector of V_1

$$v := a \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} + b \begin{pmatrix} 3 \\ -4 \\ 5 \end{pmatrix}$$

is in V_2. It must hold that $6a + 4b = 0 \implies a = -\frac{2}{3}b$. And vice-versa, if $a = -\frac{2}{3}b$ then $v \in V_2$. Therefore,

$$V_1 \cap V_2 = \left\{ b \cdot \left[-\frac{2}{3} \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} + \begin{pmatrix} 3 \\ -4 \\ 5 \end{pmatrix} \right] : b \in \mathbb{R} \right\},$$

which implies that it has dimension 1.

(b) By the above computation,

$$\left\{ v_0 := \begin{pmatrix} 7/3 \\ -16/3 \\ 3 \end{pmatrix} \right\}$$

is a basis of $V_1 \cap V_2$.

(c) Since $x + y + z = 0$ for all $(x, y, z)^T \in V_1 \cap V_2$, we immediately have $(1, 1, 1)^T \in (V_1 \cap V_2)^\perp$. Now note that for

$$u := \begin{pmatrix} 1 \\ 0 \\ -7/9 \end{pmatrix},$$

we have

$$\langle v_0, u \rangle = 0 \quad \text{and} \quad u \notin \operatorname{span}\left(\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}\right).$$

So, since $(V_1 \cap V_2)^\perp$ is two-dimensional, $\{u, (1, 1, 1)^T\}$ is a basis of $(V_1 \cap V_2)^\perp$.

(d) In (c), we have shown that

$$\langle v_0, u \rangle = 0 \text{ and that } \langle v_0, (1, 1, 1)^T \rangle = 0.$$

Now, define

$$f : \mathbb{R}^3 \to \mathbb{R}^3,\ v \mapsto \langle v, u \rangle u + \langle v, (1, 1, 1)^T \rangle (1, 1, 1)^T.$$

This map is clearly linear, and, by the above observation, we have $v_0 \in \operatorname{Ker}(f)$. Moreover,

$$f\left(\begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix}\right) = u \quad \text{and} \quad f\left(\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}\right) = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}.$$

Since $\{u, (1, 1, 1)^T\} \subset \operatorname{Im}(f)$ is a basis of $(V_1 \cap V_2)^\perp$, we have $(V_1 \cap V_2)^\perp \subseteq \operatorname{Im}(f)$. So $\dim(\operatorname{Im}(f)) \geqslant 2$. On the other hand, since $v_0 \in \operatorname{Ker}(f)$, $\dim(\operatorname{Ker}(f)) \geqslant 1$ and, by the rank-nullity formula, $\dim(\operatorname{Im}(f)) + \dim(\operatorname{Ker}(f)) = 3$. This implies that $\dim(\operatorname{Im}(f)) = 2$ and $\dim(\operatorname{Ker}(f)) = 1$. Hence, $\operatorname{Ker}(f) = \operatorname{span}(v_0) = V_1 \cap V_2$.

6. *(7 Points)* Let M be a finite non-empty set and $V := \{f : M \to \mathbb{K}\}$ be the set of all maps to $\mathbb{K}$. It is well known that V is a vector space over $\mathbb{K}$ when endowed with the following operations:

$$(f + g)(x) := f(x) + g(x),\ \forall f, g \in V, x \in M,$$
$$(af)(x) := af(x),\ \forall f \in V, a \in \mathbb{K}, x \in M.$$

(a) (2 Points) Determine the dimension of V.

(b) (3 Points) Fix an $m \in M$ and show that the set

$$U_m := \{f \in V \mid f(m) = 0\}$$

is a vector subspace of V. Additionally, compute the dimension of U_m.

(c) (2 Points) Determine a linear complement of U_m in V.

Solution: Let us denote $M = \{a_1, a_2, \dots, a_n\}$ for some integer $n \geqslant 0$.

(a) Consider the set of maps $\mathcal{B} = \left\{ f_i : M \to \mathbb{K} \mid 1 \leqslant i \leqslant n \right\}$ defined such that

$$f_i(a_j) = \begin{cases} 1, & i = j \\ 0, & i \neq j \end{cases}$$

Any map in V can be written as a $\mathbb{K}$-linear combination of the f_i's, in other words $\mathcal{B}$ is a generating set for V. Moreover, any $\mathbb{K}$-linear combination $\sum_{i=1}^{n} b_i f_i(x)$ vanishes on the whole of M if and only if all of its coefficients vanish. Hence, $\mathcal{B}$ is a basis of V and $\dim(V) = |\mathcal{B}| = n$.

(b) Clearly, the zero map belongs to U_m. Let $f, g \in U_m$ and $\alpha \in \mathbb{K}$. Using the vector space operations given above, we observe that

$$(f + \alpha g)(m) = f(m) + \alpha g(m) = 0.$$

It follows that U_m is closed under addition and scalar multiplication, which in turn shows that it is a subspace.

Up to reordering, we may assume that the m we fixed is equal to a_1. Note that U_m is at least $(n-1)$-dimensional since for all $i \in \{2, \dots, n\}$, $f_i \in U_m$. However, $f_1 \notin U_m$. Hence

$$n - 1 \leqslant \dim(U_m) < n \implies \dim(U_m) = n - 1.$$

This implies that $\mathcal{B} \setminus \{f_1\}$ is a basis for U_m.

(c) We continue to assume here that $m = a_1$. Consider $W := \langle f_1 \rangle$. Since $\mathcal{B}$ is a basis and $\{f_2, \dots, f_n\} \subseteq U_m$, we have

$$V = W + U_m.$$

Now, let $h \in W \cap U_m$. Since $h \in W$, $h = \alpha_1 f_1$ for some $\alpha_1 \in \mathbb{K}$. Moreover, by (b), there exist $\alpha_2, \dots, \alpha_n$ such that $h = \sum_{i=2}^{n} \alpha_i f_i$. It follows that

$$\alpha_1 f_1 = \sum_{i=2}^{n} \alpha_i f_i,$$

which contradicts the linear independence of $\mathcal{B}$ except if $h = 0$. It follows that $W \cap U_m = \{0\}$.

7. *(10 Points)* Consider the complex matrices

$$\sigma_0 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \quad \sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \quad B = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$$

(a) (2 Points) Show that the tuple $(\sigma_0, \sigma_1, \sigma_2, \sigma_3)$ defines a basis of the complex vector space $M_{2,2}(\mathbb{C})$ of 2×2 matrices.

(b) (1 Point) Show that the map

$$\begin{aligned} T : M_{2,2}(\mathbb{C}) &\to M_{2,2}(\mathbb{C}) \\ X &\mapsto XB - BX \end{aligned}$$

is linear.

(c) (3 Points) Compute the representation matrix of T with respect to the basis $(\sigma_0, \sigma_1, \sigma_2, \sigma_3)$.

(d) (4 Points) Determine a basis of $M_{2,2}(\mathbb{C})$ that trigonalizes T.

Solution:

(a) The space $M_{2,2}(\mathbb{C})$ is 4-dimensional over $\mathbb{C}$. Hence we just need to show that the σ_i's are linearly independent over $\mathbb{C}$. Assume that there exist $\alpha, \beta, \gamma, \delta \in \mathbb{C}$ such that

$$\alpha\sigma_0 + \beta\sigma_1 + \gamma\sigma_2 + \delta\sigma_3 = 0$$

if and only if

$$\begin{array}{rcl} \alpha + \delta &=& 0 \\ \beta - i\gamma &=& 0 \\ \beta + i\gamma &=& 0 \\ \alpha - \delta &=& 0 \end{array} \iff \begin{array}{rcl} 2\delta &=& 0 \\ \alpha &=& \delta \\ 2\beta &=& 0 \\ \gamma &=& -i\beta \end{array} \iff 0 = \alpha = \beta = \gamma = \delta.$$

This proves that the σ_i's form a basis, which we denote C.

(b) Let $M, N \in M_{2,2}(\mathbb{C})$ and $\alpha \in \mathbb{C}$. We have

$$\begin{aligned} T(M + \alpha N) = (M + \alpha N)B - B(M + \alpha N) &= MB + \alpha NB - BM - B(\alpha N) \\ &= MB - BM + \alpha(NB - BN) \\ &= T(M) + \alpha T(N). \end{aligned}$$

This shows that T is linear.

(c) We compute the image of each basis element via T. We have

$$\begin{aligned} T\sigma_0 &= 0, \\ T\sigma_1 &= \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} = -\sigma_3, \\ T\sigma_2 &= \begin{pmatrix} -i & 0 \\ 0 & i \end{pmatrix} = -i\sigma_3, \\ T\sigma_3 &= \begin{pmatrix} 0 & 2 \\ 0 & 0 \end{pmatrix} = \sigma_1 + i\sigma_2. \end{aligned}$$

It follows that

$$[T]_{\mathcal{C}}^{\mathcal{C}} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & i \\ 0 & -1 & -i & 0 \end{pmatrix}.$$

(d) The characteristic polynomial of T is $x^4 \in \mathbb{C}[x]$. Hence its single eigenvalue is $\lambda = 0$ and it has algebraic multiplicity 4. To compute a basis of $M_{2,2}(\mathbb{C})$ that trigonalizes T, we want to find a basis for the generalized eigenspace $\widetilde{Eig_T}(0)$. We have

$$([T]_{\mathcal{C}}^{\mathcal{C}})^2 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & -1 & -i & 0 \\ 0 & -i & 1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad ([T]_{\mathcal{C}}^{\mathcal{C}})^3 = 0.$$

So,

$$\{0\} \subseteq \operatorname{Ker}([T]_{\mathcal{C}}^{\mathcal{C}}) \subseteq \operatorname{Ker}(([T]_{\mathcal{C}}^{\mathcal{C}})^2) \subsetneq \operatorname{Ker}(([T]_{\mathcal{C}}^{\mathcal{C}})^3) = M_{2,2}(\mathbb{C}).$$

We compute that $(1,0,0,0)^T, (0,0,0,1)^T$ generate $\operatorname{Ker}(([T]_{\mathcal{C}}^{\mathcal{C}})^2)$. It follows that

$$v_0 := \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix} \in \operatorname{Ker}(([T]_{\mathcal{C}}^{\mathcal{C}})^3) \setminus \operatorname{Ker}(([T]_{\mathcal{C}}^{\mathcal{C}})^2)$$

$$\implies \begin{pmatrix} 0 \\ 0 \\ 0 \\ -1 \end{pmatrix} = Av_0 \in \operatorname{Ker}(([T]_{\mathcal{C}}^{\mathcal{C}})^2) \setminus \operatorname{Ker}([T]_{\mathcal{C}}^{\mathcal{C}})$$

$$\implies \begin{pmatrix} 0 \\ -1 \\ -i \\ 0 \end{pmatrix} = A^2 v_0 \in \operatorname{Ker}([T]_{\mathcal{C}}^{\mathcal{C}}).$$

To complete our basis, we choose $(1,0,0,0)^T \in \operatorname{Ker}([T]_{\mathcal{C}}^{\mathcal{C}})$. The change of basis matrix

$$\begin{pmatrix} 0 & 0 & 0 & 1 \\ -1 & 0 & 1 & 0 \\ -i & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix}$$

trigonalizes T.

8. *(11 Points)* Let f be an endomorphism of a finite-dimensional unitary vector space $(V, \langle\cdot,\cdot\rangle)$ over $\mathbb{C}$. Assume that $f^n = id_V$ for some $n \geqslant 1$.

(a) (2 Points) Show that the expression

$$\langle\langle v, w\rangle\rangle := \sum_{i=0}^{n-1} \langle f^i(v), f^i(w)\rangle$$

defines (a possibly different) inner product $\langle\langle\cdot,\cdot\rangle\rangle$ on V.

(b) (2 Points) Which properties does f have with respect to $\langle\langle\cdot,\cdot\rangle\rangle$? Deduce from this that f is diagonalisable.

(c) (2 Points) Show again, using the minimal polynomial, that f is diagonalisable.

(d) (5 Points) Consider the case $V = \mathbb{C}^2$ with the standard inner product. Determine $\langle\langle\cdot,\cdot\rangle\rangle$ and an orthonormal basis of eigenvectors for the endomorphism $f : v \mapsto Av$ where

$$A = \begin{pmatrix} 0 & 1 \\ -1 & 1 \end{pmatrix}.$$

Solution:

(a) Let $v_1, v_2, w \in V$, and let $\alpha \in \mathbb{C}$. We have

$$\begin{aligned}
\langle\langle v_1 + \alpha v_2, w\rangle\rangle &= \sum_{i=0}^{n-1} \langle f^i(v_1 + \alpha v_2), f^i(w)\rangle \\
&= \sum_{i=0}^{n-1} \langle f^i(v_1) + \alpha f^i(v_2), f^i(w)\rangle \\
&= \sum_{i=0}^{n-1} \langle f^i(v_1), f^i(w)\rangle + \alpha \sum_{i=0}^{n-1} \langle f^i(v_2), f^i(w)\rangle \\
&= \langle\langle v_1, w\rangle\rangle + \alpha\langle\langle v_2, w\rangle\rangle.
\end{aligned}$$

We used the facts that f^i is linear for $0 \leqslant i \leqslant n-1$, and that the standard hermitian product is linear in the first variable to obtain the chain of equalities above. Similarly, we use linearity of f^i and sesquilinearity of the standard hermitian product in the second variable to prove sequilinearity of the expression in the second variable. By the hermitian property of the standard hermitian product and by the anti-linearity of complex conjugation, we obtain that the exression is hermitian, i.e.

$$\forall v, w \in V : \quad \langle\langle v, w\rangle\rangle = \overline{\langle\langle w, v\rangle\rangle}.$$

Finally, let $v \in V \setminus \{0\}$. We have

$$\langle\langle v, v\rangle\rangle = \sum_{i=0}^{n-1} \langle f^i(v), f^i(v)\rangle = \langle v, v\rangle + \sum_{i=1}^{n-1} \langle f^i(v), f^i(v)\rangle.$$

By positivity of the standard hermitian product, the first term above is strictly positive and the remaining terms in the sum are non-negative. Hence,

$$\langle\langle v, v\rangle\rangle > 0.$$

(b) We observe that

$$\begin{aligned}
\langle\langle f(v), w\rangle\rangle &= \sum_{i=0}^{n-1}\langle f^{i+1}(v), f^{i}(w)\rangle \\
&= \sum_{i=1}^{n}\langle f^{i}(v), f^{i-1}(w)\rangle \\
&= \sum_{i=0}^{n-1}\langle f^{i}(v), f^{i}(f^{n-1}(w))\rangle \\
&= \langle\langle v, f^{n-1}(w)\rangle\rangle,
\end{aligned}$$

where we used that $f^n = id_V \implies f^{i-1} = f^{i+n-1}$ for all $i \in \{0, \dots, n-1\}$. This implies that $f^* = f^{n-1} = f^{-1}$ with respect to $\langle\langle\cdot,\cdot\rangle\rangle$. This implies that f is an isometry, so that it is normal. Hence f is diagonalisable by the Spectral Theorem over $\mathbb{C}$. Indeed, we have

$$\langle\langle f(v), f(w)\rangle\rangle = \langle\langle v, f^{*}\circ f(w)\rangle\rangle = \langle\langle v, w\rangle\rangle$$

and

$$f\circ f^{-1} = id_V = f^{-1}\circ f.$$

(c) Let n_0 be the smallest positive integer n such that $f^{n_0} = id_V$. Then f is a root of the polynomial

$$x^{n_0} - 1 \in \mathbb{C}[x].$$

This polynomial factors as

$$(x-1)(x^{n_0-1} + x^{n_0-2} + \cdots + 1) = \prod_{k=0}^{n_0-1}(x - \zeta^k),$$

where $\zeta = e^{2\pi i/n_0} \in \mathbb{C}$. Hence the minimal polynomial of f, which must divide the above polynomial, factors into distinct linear factors over $\mathbb{C}$. It follows that f is diagonalisable.

(d) First of all, we compute that $f^6 = id_V$ and that 6 is the smallest integer to satisfy this property. We write out the definition of $((\cdot,\cdot))$ and compute that it is given by the hermitian matrix

$$\begin{pmatrix} 8 & -4 \\ -4 & 8 \end{pmatrix}.$$

The characteristic polynomial of f is given by

$$x^2 - x + 1 \in \mathbb{C}[x].$$

Hence the eigenvalue-eigenvector pairs of f are

$$\left(\lambda_1 = \frac{1+i\sqrt{3}}{2},\ v_1 = \begin{pmatrix} 1 \\ \lambda_1 \end{pmatrix}\right), \quad \left(\lambda_2 = \frac{1-i\sqrt{3}}{2},\ v_2 = \begin{pmatrix} 1 \\ \lambda_2 \end{pmatrix}\right).$$

We compute that

$$((v_2, v_1)) = \begin{pmatrix} 1 & \lambda_2 \end{pmatrix} \begin{pmatrix} 8 & -4 \\ -4 & 8 \end{pmatrix} \begin{pmatrix} 1 \\ \overline{\lambda_1} \end{pmatrix} = 8(1 - \lambda_2 + \lambda_2^2) = 0.$$

Here we used that $\overline{\lambda_1} = \lambda_2$ and that λ_2 is a root of $x^2 - x + 1$. So, our basis of eigenvectors is already orthogonal with respect to $\langle\langle \cdot, \cdot \rangle\rangle$. We only need to normalize each eigenvector. We compute that $\langle\langle v_1, v_1 \rangle\rangle = 12 = \langle\langle v_2, v_2 \rangle\rangle$. It follows that

$$\left\{ \frac{1}{2\sqrt{3}} v_1, \frac{1}{2\sqrt{3}} v_2 \right\}$$

is an orthonormal basis of eigenvectors of f with respect to $\langle\langle \cdot, \cdot \rangle\rangle$.

9. *(5 Points)* Let U and V be finite dimensional vector spaces over a field $\mathbb{K}$. Let $f \in End(U)$ and $g \in End(V)$, and consider the endomorphism $f \otimes g : U \otimes V \to U \otimes V$. Express Trace($f \otimes g$) in terms of Trace(f) and Trace(g).

   Solution: Let $\mathcal{B} = \{u_1, \dots, u_m\}$, respectively $\mathcal{C} = \{v_1, \dots, v_n\}$, be a basis of U, respectively V. Then, a basis for the tensor product $U \otimes V$ is given by

   $$\mathcal{D} = \{u_i \otimes v_k \mid 1 \leqslant i \leqslant m,\ 1 \leqslant k \leqslant n\}.$$

   Let $A = (a_{ij})_{i,j=1,\dots,m} = [f]_{\mathcal{B}}^{\mathcal{B}}$ and $B = (b_{k\ell})_{k,\ell=1,\dots,n} = [g]_{\mathcal{C}}^{\mathcal{C}}$. Then, by definition

   $$\begin{aligned}(f \otimes g)(u_i \otimes v_k) &= f(u_i) \otimes g(v_k) \\ &= \left(\sum_{j=1}^{m} a_{ji} u_j\right) \otimes \left(\sum_{\ell=1}^{n} b_{\ell k} v_\ell\right) \\ &= \sum_{j=1}^{m} \sum_{\ell=1}^{n} a_{ji} b_{\ell k} (u_j \otimes v_\ell).\end{aligned}$$

   So, the diagonal entries of $[f \otimes g]_{\mathcal{D}}^{\mathcal{D}}$ are given by

   $$\{a_{ii} b_{kk} \mid 1 \leqslant i \leqslant m,\ 1 \leqslant k \leqslant n\}.$$

   Since the trace does not depend on a choice of basis, we obtain

   $$\mathrm{tr}(f \otimes g) = \sum_{i=1}^{m} \sum_{k=1}^{n} a_{ii} b_{kk} = \left(\sum_{i=1}^{m} a_{ii}\right)\left(\sum_{k=1}^{n} b_{kk}\right) = \mathrm{tr}(f)\mathrm{tr}(g).$$