Lineare Algebra I/II - Mock exam

1. *(5 points)*

   (a) (2 points) Give the definition of the symmetric group on n-elements S_n. For a permutation $\sigma \in S_n$ give the defintion of $\mathrm{sign}(\sigma)$. You do not have to show that $\mathrm{sign}(\sigma)$ is well defined.

   (b) (3 points) Let $\sigma \in S_n$ and let $A \in M_{n\times n}(K)$ be the matrix obtained from the identity matrix I_n by permuting its rows using the permutation σ. Prove that $\det(A) = \mathrm{sign}(\sigma)$.

2. *(5 points)* Let V be a finite dimensional vector space over the field $\mathbb{K}$. Let $T \in End(V)$ and let $\lambda_1, \lambda_2 \in \mathbb{K}$ be two distinct eigenvalues of T. Denote by $Eig_T(\lambda_i)$ the eigenspace of λ_i and by $\widetilde{Eig}_T(\lambda_i)$ the generalized eigenspace of λ_i, $i = 1, 2$.

   (a) (2 points) Prove that $Eig_T(\lambda_1) \cap Eig_T(\lambda_2) = \{0\}$.

   (b) (3 points) Assume that the characteristic polynomial $p_T(x)$ of T splits as a product of linear factors in $\mathbb{K}[x]$. Prove that $\widetilde{Eig}_T(\lambda_1) \cap \widetilde{Eig}_T(\lambda_2) = \{0\}$.

3. *(10 Points)* For each statement, mark with a cross whether it is true (T) or false (F). Correct answers are awarded +1 point, incorrect answers or no answer 0 points.

   (a) □T □F The matrix $B = \begin{pmatrix} 0 & \sqrt{3}i & 0 \\ \sqrt{3}i & 0 & 0 \\ 0 & 0 & -3 \end{pmatrix}$ is diagonalisable over $\mathbb{C}$.

   (b) □T □F Every diagonalisable matrix $A \in M_{n,n}(\mathbb{K})$ consists of n linearly independent column vectors.

   (c) □T □F Every matrix $A \in M_{n,n}(\mathbb{R})$, whose eigenvalues are all positive, is symmetric.

   (d) □T □F Every matrix $A \in \mathrm{SO}(3)$ satisfies $\mathrm{tr}(A) \leqslant 3$.

   (e) □T □F Let $A \in M_{4,4}(\mathbb{C})$ be a matrix with characteristic polynomial $p_A(x) = (x+i)^2(x-\sqrt{2})(x+2)$. Then A is diagonalisable if and only if $\dim(\mathrm{Ker}(A + i1_4)) = 2$.

   (f) □T □F Let f be a vector space endomorphism. Then, for every eigenvalue of f there exists a unique eigenvector.

   (g) □T □F Let V be a finite-dimensional vector space, and let V^* be its dual space. Then $V \cong V^*$.

(h) □T □F Let V be the vector space of continuous functions $f : \mathbb{R} \to \mathbb{R}$ such that $f(x+1) = f(x)$ for all $x \in \mathbb{R}$. The map

$$(f, g) = \int_0^1 f(x)g(x + \frac{1}{2})dx$$

defines a scalar product over V.

(i) □T □F Consider $v_1 \in \mathbb{R}^3$ with $\|v_1\| = 1$. There is exactly one vector $v \in \mathbb{R}^3$, such that $(v, v_1) = 1$ and $\|v\| = 1$.

(j) □T □F Consider two endomorphisms f, g of a finite-dimensional euclidian vector space. It holds that

$$f^* g^* = gf \Leftrightarrow fg = g^* f^*.$$

4. *(14 Points)* Write your answer directly on the exam sheet. You do not have to justify your answer.

(a) Compute the determinant of

$$A_\lambda = \begin{pmatrix} 1 & -\lambda & \lambda^2 & -\lambda^3 \\ 1 & 2\lambda & 4\lambda^2 & 8\lambda^3 \\ 1 & \lambda & \lambda^2 & \lambda^3 \\ 1 & -2\lambda & 4\lambda^2 & -8\lambda^3 \end{pmatrix} \quad \text{für } \lambda \in \mathbb{C}. \qquad \det(A_\lambda) = \boxed{}$$

(b) For which $\alpha \in \mathbb{C}$ (give all of them) is the matrix

$$A_\alpha = \begin{pmatrix} 1 & -1 & 0 \\ \alpha & -1 & 0 \\ 2 & 1 & \alpha \end{pmatrix} \in M_{3,3}(\mathbb{C}) \text{ invertible?} \qquad \alpha \in \boxed{}$$

(c) Let A_α be the same as in (b), and let α be such that A_α is invertible. Compute the inverse of A_α. (Your answer will depend on the variable α.)

$$A_\alpha^{-1} = \boxed{}$$

(d) Compute the eigenvalues of the matrix

$$\begin{pmatrix} 0 & -4 & 6 \\ -3 & 5 & 0 \\ 2 & -4 & 1 \end{pmatrix} \in M_{3,3}(\mathbb{R}). \qquad \text{Antwort: } \boxed{}$$

(e) Let $A = \begin{pmatrix} 1 & 0 & 3 \\ -1 & \alpha & -2 \\ 2 & -\alpha & 2 \end{pmatrix}$ and $b = \begin{pmatrix} -3 \\ 1 \\ 2 \end{pmatrix}$. Find all $x \in \mathbb{R}^3$ such that $Ax = b$, for $\alpha \in \mathbb{R} \setminus \{0\}$.

$$x \in \boxed{}$$

(f) Compute the minimal polynomial of the matrix

$$A = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 1 & -1 & 1 \end{pmatrix}.$$

Antwort: [ ]

(g) Let $T : \mathbb{R}^2 \to \mathbb{R}[x]$ be the unique linear map such that $T(5,2) = 11 + 22x$ and $T(1,7) = 33 - 11x$. Compute $T(1,4)$.

$T(1,4) =$ [ ]

5. *(10 Points)* Consider the vector subspaces

$$V_1 = \left\langle \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}, \begin{pmatrix} 3 \\ -4 \\ 5 \end{pmatrix} \right\rangle$$

and

$$V_2 = \left\{ \begin{pmatrix} x \\ y \\ z \end{pmatrix} \in \mathbb{R}^3 \middle| x + y + z = 0 \right\}$$

in $V = \mathbb{R}^3$.

(a) (2 Points) Determine the dimension of V_1, V_2 and $V_1 \cap V_2$.

(b) (3 Points) Find a basis of $V_1 \cap V_2$.

(c) (3 Points) Find a basis of the orthogonal complement of $V_1 \cap V_2$ with respect to the standard scalar product over $V = \mathbb{R}^3$.

(d) (2 Points) Find a linear map $f : \mathbb{R}^3 \to \mathbb{R}^3$ with $\mathrm{Ker}(f) = V_1 \cap V_2$.

6. *(7 Points)* Let M be a finite non-empty set and $V := \{f : M \to \mathbb{K}\}$ be the set of all maps to $\mathbb{K}$. It is well known that V is a vector space over $\mathbb{K}$ when endowed with the following operations:

$$\begin{aligned}(f+g)(x) &:= f(x)+g(x),\ \forall f,g \in V, x \in M,\\ (af)(x) &:= af(x),\ \forall f \in V, a \in \mathbb{K}, x \in M.\end{aligned}$$

   (a) (2 Points) Determine the dimension of V.

   (b) (3 Points) Fix an $m \in M$ and show that the set

$$U_m := \{f \in V \mid f(m) = 0\}$$

   is a vector subspace of V. Additionally, compute the dimension of U_m.

   (c) (2 Points) Determine a linear complement of U_m in V.

7. *(10 Points)* Consider the complex matrices

$$\sigma_0 = \begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix} \quad \sigma_1 = \begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix} \quad \sigma_2 = \begin{pmatrix}0 & -i\\ i & 0\end{pmatrix} \quad \sigma_3 = \begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix} \quad B = \begin{pmatrix}1 & 1\\ 0 & 1\end{pmatrix}$$

   (a) (2 Points) Show that the tuple $(\sigma_0, \sigma_1, \sigma_2, \sigma_3)$ defines a basis of the complex vector space $M_{2,2}(\mathbb{C})$ of 2×2 matrices.

   (b) (1 Point) Show that the map

$$\begin{aligned}T : M_{2,2}(\mathbb{C}) &\to M_{2,2}(\mathbb{C})\\ X &\mapsto XB - BX\end{aligned}$$

   is linear.

   (c) (3 Points) Compute the representation matrix of T with respect to the basis $(\sigma_0, \sigma_1, \sigma_2, \sigma_3)$.

   (d) (4 Points) Determine a basis of $M_{2,2}(\mathbb{C})$ that trigonalizes T.

8. *(11 Points)* Let f be an endomorphism of a finite-dimensional unitary vector space $(V, \langle\cdot,\cdot\rangle)$ over $\mathbb{C}$. Assume that $f^n = id_V$ for some $n \geqslant 1$.

   (a) (2 Points) Show that the expression

   $$\langle\langle v, w\rangle\rangle := \sum_{i=0}^{n-1} \langle f^i(v), f^i(w)\rangle$$

   defines (a possibly different) inner product $\langle\langle\cdot,\cdot\rangle\rangle$ on V.

   (b) (2 Points) Which properties does f have with respect to $\langle\langle\cdot,\cdot\rangle\rangle$? Deduce from this that f is diagonalisable.

   (c) (2 Points) Show again, using the minimal polynomial, that f is diagonalisable.

   (d) (5 Points) Consider the case $V = \mathbb{C}^2$ with the standard inner product. Determine $\langle\langle\cdot,\cdot\rangle\rangle$ and an orthonormal basis of eigenvectors for the endomorphism $f : v \mapsto Av$ where

   $$A = \begin{pmatrix} 0 & 1 \\ -1 & 1 \end{pmatrix}.$$

9. *(5 Points)* Let U and V be finite dimensional vector spaces over a field $\mathbb{K}$. Let $f \in End(U)$ and $g \in End(V)$, and consider the endomorphism $f \otimes g : U \otimes V \to U \otimes V$. Express Trace($f \otimes g$) in terms of Trace(f) and Trace(g).