Problems and suggested solution

Question 1

Determine whether each of the following statements is true or false.
Throughout this exercise, we let V, W, and W' be vector spaces over a field K, and let $U \subseteq V$ be a subspace of V.

1.MC1 [**1 Point**] Let W_1 and W_2 be linear complements of U in V. Then, W_1 is isomorphic to W_2.

(A) True

(B) False

Solution:

True.

1.MC2 [**1 Point**] Let $f : V \to W$ and $g : W \to W'$ be linear maps. Then $g \circ f \equiv 0$ implies $f \equiv 0 \vee g \equiv 0$.

(A) True

(B) False

1.MC3 [**1 Point**] Consider a subspace $V' \subseteq V$ such that $U \subseteq V'$. Then

$$\left. {V / U} \middle/ {V' / U} \right. \cong \left. V \middle/ V' \right. .$$

(A) True

(B) False

Solution:

True.

1.MC4 [**1 Point**] The matrix

$$\begin{pmatrix} 2 & 0 \\ 2 & 3 \end{pmatrix}$$

is invertible.

(A) True

(B) False

Solution:

True.

1.MC5 [1 Point] The set

$$\{1, x+1, (x+1)^2, (x+1)^3\} \subset K[x]$$

is a basis for $K[x]_3$, the space of polynomials over K of degree at most 3.

(A) True

(B) False

Solution:

True.

1.MC6 [1 Point] Let $V_1, V_2 \subseteq V$ be subspaces. We have

$$U + (V_1 \cap V_2) = (U + V_1) \cap (U + V_2).$$

(A) True

(B) False

Solution:

False.

1.MC7 [1 Point] The set

$$\{(t,0,1),(0,t^2,1),(1,0,t)\} \subset \mathbb{R}^3$$

is linearly independent for all $t \in \mathbb{R}$.

(A) True

(B) False

Solution:

False.

1.MC8 [1 Point] Let $f : V \to W$ and $g : W \to W'$ be linear maps. Assume that f is surjective. We have

$$\operatorname{rank}(g \circ f) = \operatorname{rank}(g).$$

(A) True

(B) False

Solution:

True.

1.MC9 [1 Point] Let $f : V \to W$ be an injective linear map. Then its dual map $f^* : W^* \to V^*$ is also injective.

(A) True

(B) False

Solution:

False.

1.MC10 [1 Point] Consider a linearly independent subset $\{v_1, v_2, \cdots, v_m\} \subset V$, for some natural number $m \geq 1$, and let $w \in V$. Then

$$\dim \operatorname{Sp}(v_1 + w, v_2 + w, \cdots, v_m + w) < m.$$

(A) True

(B) False

Solution:

False.

Question 2

Let V, W be vector spaces over a field K.

2.Q1 [1 Point] Give the definition of $\mathrm{Hom}(V, W)$.

2.Q2 [9 Points] Fix a subspace $U \subseteq V$ and consider the space

$$H := \{f \in \mathrm{Hom}(V, W) \mid f|_U \equiv 0\}.$$

Show that H is isomorphic to $\mathrm{Hom}(V/U, W)$.

Remark. None of the spaces above are assumed to be finite-dimensional.

Solution:

1. By definition,
$$\mathrm{Hom}(V, W) = \{f : V \to W \mid f \text{ is a linear map}\}.$$

2. Denote $[v]$ the image of a vector $v \in V$ via the quotient map $V \to V/U$. Consider the map

$$\begin{array}{rccc} \phi : & H & \to & \mathrm{Hom}(V/U, W) \\ & f & \mapsto & \phi(f) : [v] \mapsto f(v) \end{array}$$

The map is well-defined since if $[v] = [v']$ in V/U, then there exists $u \in U$ such that

$$f(v') = f(v + u) = f(v) + f(u) = f(v),$$

as $f \in H$. We note that $\ker(\phi) = \{f \in H \mid \forall\, v \in V : f(v) = 0\}$. Hence ϕ is injective.

Let $g : V/U \to W$ be a linear map. We consider the map

$$\begin{array}{rccc} \tilde{g} : & V & \to & W \\ & v & \mapsto & g([v]) \end{array}$$

By definition, $\tilde{g}(u) = g([0]) = 0$ for all $u \in U$. Hence $\tilde{g} \in H$ is a preimage of g through ϕ. This shows that ϕ is surjective.

To conclude that it is an isomorphism, we still need to show that ϕ is linear. Let $\alpha \in K$ and $f, g \in H$. For all $v \in V$, we have

$$\phi(\alpha f + g)([v]) = (\alpha f + g)(v) = \alpha f(v) + g(v) = \alpha\phi(f)([v]) + \phi(g)([v]).$$

Therefore $\phi(\alpha f + g) = \alpha\phi(f) + \phi(g)$ in $\mathrm{Hom}(V, W)$, which shows that ϕ is linear and concludes the proof.

Question 3

Let K be a field and denote $K[x]_3$ the space of polynomials with coefficients in K of degree at most 3. Fix some $a \in K$ and consider the linear maps

$$\begin{array}{rccc} \mathrm{Comp}: & K[x]_3 & \to & K[x]_3 \\ & p(x) & \mapsto & p(ax) \end{array}$$

and

$$\begin{array}{rccc} D: & K[x]_3 & \to & K[x]_3 \\ & p(x) & \mapsto & p'(x) \end{array}$$

3.Q1 [6 Points] Write the matrices representing Comp and D, respectively, with respect to the standard basis for $K[x]_3$.

3.Q2 [4 Points] Write the matrix representing $D \circ \mathrm{Comp}$ with respect to the standard basis of $K[x]_3$.

Solution:

1. Let us denote $\mathcal{B}$ the standard basis of $K[x]_3$. From $\mathrm{Comp}(x^k) = a^k x^k$, we deduce that

$$[\mathrm{Comp}]_{\mathcal{B}}^{\mathcal{B}} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & a & 0 & 0 \\ 0 & 0 & a^2 & 0 \\ 0 & 0 & 0 & a^3 \end{pmatrix}.$$

From $D(x^k) = kx^{k-1}$, we deduce that

$$[D]_{\mathcal{B}}^{\mathcal{B}} = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 3 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

2. You have seen that this amounts to computing the product of matrices $[D]_{\mathcal{B}}^{\mathcal{B}} \cdot [\mathrm{Comp}]_{\mathcal{B}}^{\mathcal{B}}$. We have

$$\begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 3 \\ 0 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & a & 0 & 0 \\ 0 & 0 & a^2 & 0 \\ 0 & 0 & 0 & a^3 \end{pmatrix} = \begin{pmatrix} 0 & a & 0 & 0 \\ 0 & 0 & 2a^2 & 0 \\ 0 & 0 & 0 & 3a^3 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

Question 4

Let V and W be vector spaces over a field K. Consider a linear map $f : V \to W$.

4.Q1 [2 Points] Define the rank of f.

4.Q2 [8 Points] Assume that V is finite-dimensional. State and prove the formula relating the dimension of V to the dimension of the kernel of f and the rank of f.

Solution:

1. The expected definition is
$$\operatorname{rank}(f) = \dim(\operatorname{Im}(f)).$$

2. We state the rank-nullity theorem. For the linear map of K-vector spaces $f : V \to W$, with V finite-dimensional, we have
$$\dim V = \operatorname{rank}(f) + \dim(\ker(f)).$$
It can be proved as follows: let n denote the dimension of V and let $\{u_1, \ldots, u_r\}$ be a basis of $\ker(f)$. It can be extended to a basis of V, which we denote $B = \{u_1, \ldots, u_r, v_1, \ldots, v_{n-r}\}$. Since $f(u_i) = 0$ for $i = 1, \ldots, r$, we have
$$\operatorname{Im}(f) = \operatorname{Sp}(f(u_1), \ldots, f(u_r), f(v_1), \ldots, f(v_{n-r})) = \operatorname{Sp}(f(v_1), \ldots, f(v_{n-r})).$$
We now prove that $\{f(v_1), \ldots, f(v_{n-r})\}$ is a basis of $\operatorname{Im}(f)$ by showing that it is a linearly independent set. We assume for a contradiction that it isn't. Then there exist coefficients $\{a_1, \ldots, a_{n-r}\} \subset K$, not all vanishing, such that
$$a_1 f(v_1) + \cdots + a_{n-r} f(v_{n-r}) = 0.$$
This is equivalent to
$$\sum_{i=1}^{n-r} f(a_i v_i) = 0 \Leftrightarrow f\left(\sum_{i=1}^{n-r} a_i v_i\right) = 0,$$
which implies that $\sum_{i=1}^{n-r} a_i v_i \in \ker(f) = \operatorname{Sp}(u_1, \ldots, u_r)$. This yields a contradiction to the fact that B is a basis. So $\{f(v_1), \ldots, f(v_{n-r})\}$ is a basis of $\operatorname{Im}(f)$ and thereupon, $\operatorname{rank}(f) = n - r$.
We conclude that
$$\dim(\ker(f)) + \operatorname{rank}(f) = r + n - r = n = \dim V.$$

Question 5

Let V be a vector space over a field K.

5.Q1 [**2 Points**] Give the definition of a basis of V.

5.Q2 [**8 Points**] Assume that V is finite-dimensional. Let S be a generating set for V and let $T \subseteq S$ be a finite linearly independent set. Write a short algorithm or explain in a few sentences how we can obtain a basis $\mathcal{B}$ of V that verifies $T \subseteq \mathcal{B} \subseteq S$.

Solution:

1. A basis of V is a linearly independent set of vectors that spans V.

2. Let r denote the cardinality of T and let u_i, $i = 1, \ldots, r$, denote the elements of T. If T spans V, we are done since T is a basis, by definition. Else, repeat the following until T has been extended to span V.

   Since $\mathrm{Sp}(T) \subsetneq V$, there exists a vector $v_1 \in S \setminus T$. Check if $v_1 \in \mathrm{Sp}(T)$. If it is, replace S with $S \setminus \{v_1\}$. If v_1 isn't spanned by T, denote T_1 the union $T \cup \{v_1\}$ and replace T with T_1.

   Since V is finite-dimensional, this algorithm will return a finite set $\mathcal{B}$ whose elements we denote $\{u_1, \ldots, u_r, w_1, \ldots, w_{n-r}\}$. By construction, $T \subset \mathcal{B} \subset S$. Moreover, $\mathcal{B}$ is linearly independent and spans V, by construction, so $\mathcal{B}$ is a basis of V.

Question 6

Consider the matrix

$$A = \begin{pmatrix} 1 & 1 & 1 \\ 2 & \alpha & 3 \\ 3 & \alpha & 2 \end{pmatrix} \in M_{3\times 3}(\mathbb{R}), \quad \alpha \in \mathbb{R}.$$

6.Q1 [6 Points] For which values of α is the matrix A invertible?

6.Q2 [4 Points] When A is not invertible, give a basis for the kernel of the linear map

$$\begin{array}{rccc} T_A : & \mathbb{R}^3 & \to & \mathbb{R}^3 \\ & v & \mapsto & A \cdot v \end{array}$$

Solution:

1. We compute an inverse as follows:

$$\left(\begin{array}{ccc|ccc} 1 & 1 & 1 & 1 & 0 & 0 \\ 2 & \alpha & 3 & 0 & 1 & 0 \\ 3 & \alpha & 2 & 0 & 0 & 1 \end{array}\right) \to \left(\begin{array}{ccc|ccc} 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & \alpha - 2 & 3 & -2 & 1 & 0 \\ 0 & \alpha - 3 & 2 & -3 & 0 & 1 \end{array}\right) \to \cdots \to \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & \frac{\alpha}{2\alpha-5} & \frac{2-\alpha}{2\alpha-5} & \frac{\alpha-3}{2\alpha-5} \\ 0 & 1 & 0 & \frac{-5}{2\alpha-5} & \frac{1}{2\alpha-5} & \frac{1}{2\alpha-5} \\ 0 & 0 & 1 & \frac{\alpha}{2\alpha-5} & \frac{\alpha-3}{2\alpha-5} & \frac{2-\alpha}{2\alpha-5} \end{array}\right)$$

Hence, A is invertible for $\alpha \in \mathbb{R} \setminus \{5/2\}$ and its inverse is given by

$$A^{-1} = \frac{1}{2\alpha - 5} \begin{pmatrix} \alpha & 2 - \alpha & \alpha - 3 \\ -5 & 1 & 1 \\ \alpha & \alpha - 3 & 2 - \alpha \end{pmatrix}.$$

2. We now let $\alpha = 5/2$. We want to find a basis for the space

$$\ker T_A = \{x \in \mathbb{R}^3 \mid A \cdot x = 0\}.$$

Let us first compute the space of solutions of $A \cdot x = 0$. In order to do this, we perform row-reduction on A. This yields

$$\left(\begin{array}{ccc|c} 1 & 1 & 1 & 0 \\ 2 & \frac{5}{2} & 3 & 0 \\ 3 & \frac{5}{2} & 2 & 0 \end{array}\right) \to \left(\begin{array}{ccc|c} 1 & 1 & 1 & 0 \\ 0 & \frac{1}{2} & 1 & 0 \\ 0 & -\frac{1}{2} & -1 & 0 \end{array}\right) \to \left(\begin{array}{ccc|c} 1 & 0 & -1 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right).$$

So,

$$\ker T_A = \{(x, y, z) \in \mathbb{R}^3 \mid x - z = 0 \wedge y + 2z = 0\}.$$

This is a one-dimensional subspace of $\mathbb{R}^3$. A basis is given by $\{(1, -2, 1)\}$, for example.