Prof. M. Einsiedler
Prof. P. Biran

Musterlösung Serie 14

Determinant

1. Compute the determinant of

$$B = \begin{pmatrix} 1 & 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 2 & 3 \\ 0 & 0 & 2 & 3 & 0 \\ 0 & 2 & 3 & 0 & 0 \\ 2 & 3 & 0 & 0 & 0 \end{pmatrix}$$

over $\mathbb{R}$ and over $\mathbb{F}_5$. Is B invertible?

Solution: We use Laplace expansion for the first column and obtain.

$$\det(B) = 1 \cdot \det \begin{pmatrix} 0 & 0 & 2 & 3 \\ 0 & 2 & 3 & 0 \\ 2 & 3 & 0 & 0 \\ 3 & 0 & 0 & 0 \end{pmatrix} + 2 \cdot \det \begin{pmatrix} 1 & 1 & 1 & 1 \\ 0 & 0 & 2 & 3 \\ 0 & 2 & 3 & 0 \\ 2 & 3 & 0 & 0 \end{pmatrix}.$$

We compute the left determinant by expanding the last row and obtain $(-3)\cdot(-27)$ and we compute the right determinant by expanding after the first column, which yields $1 \cdot (-27) - 2 \cdot (6 - 4 + 9)$. Together, this yields

$$\det(B) = 81 + 2(-27 + 14) = 55.$$

Therefore, the matrix B is invertible over $\mathbb{R}$ but not over $\mathbb{F}_5$.

2. Each of the following expressions defines a function D on the set of 3×3 matrices over the field of real numbers. In which of these cases is D a 3-linear function?

   (a) $D_1(A) = A_{11} + A_{22} + A_{33}$;

   (b) $D_2(A) = A_{11}^2 + 3A_{11}A_{22}$;

   (c) $D_3(A) = A_{11}A_{12}A_{33}$;

   (d) $D_4(A) = A_{13}A_{22}A_{32} + 5A_{12}A_{22}A_{32}$;

   (e) $D_5(A) = 0$;

   (f) $D_6(A) = 1$.

Solution: Let

$$B = \begin{pmatrix} 1 & 1 & \\ & 1 & \\ & & 1 \end{pmatrix}, \quad C = \begin{pmatrix} 2 & 2 & \\ & 1 & \\ & & 1 \end{pmatrix}.$$

These matrices will give us counterexamples for (a), (b), (c) and (f). Indeed, for (a),

$$D_1(C) = 4 \neq 6 = 2(1+1+1) = 2D_1(B).$$

For (b),

$$D_2(C) = 10 \neq 8 = 2(1 + 3 \cdot 1 \cdot 1) = 2D_2(B).$$

For (c),

$$D_3(C) = 4 \neq 2 \cdot 1 \cdot 1 \cdot 1 = 2D_3(B).$$

Finally, for (f),

$$D_6(C) = 1 \neq 2 = 2D_6(B).$$

We deduce that in these 4 cases, the map is **not** 3-linear.

We now denote the rows of a arbitrary matrix A by R_i, $i = 1, 2, 3$ and view $D(A) = D(R_1, R_2, R_3)$ as a function on the rows of 3×3 real matrices. Let $\alpha = (\alpha_i)_{i=1}^n$ be a row vector and let $\lambda \in \mathbb{R}$. Then, in the case of (d),

$$\begin{aligned} D_4(\lambda R_1 + \alpha, R_2, R_3) &= (\lambda A_{13} + \alpha_3)A_{22}A_{32} + 5(\lambda A_{12} + \alpha_2)A_{22}A_{32} \\ &= \lambda(A_{13}A_{22}A_{32} + 5A_{12}A_{22}A_{32}) + (\alpha_3 A_{22}A_{32} + 5\alpha_3 A_{22}A_{32}) \\ &= \lambda D_4(R_1, R_2, R_3) + D_4(\alpha, R_2, R_3). \end{aligned}$$

This shows that D_4 is linear in the first row and similar computations prove that D_4 is linear in the second and third row, hence that it is 3-linear.

Finally, for (e),

$$D_5(\lambda R_1 + \alpha, R_2, R_3) = 0 = \lambda \cdot 0 + 0 = \lambda D_5(R_1, R_2, R_3) + D_5(\alpha, R_2, R_3)$$

and similarly for rows 2 and 3. This shows that D_5 is 3-linear.

3. (a) Let K be a field, let $\lambda \in K$, and let $A \in M_{n\times n}(K)$. Show that:

   i. For B such that $A \xrightarrow{\lambda L_i \to L_i} B$, $\det B = \lambda \det A$;

   ii. For B such that $A \xrightarrow{L_i \leftrightarrow L_j} B$, $\det B = -\det A$;

   iii. For B such that $A \xrightarrow{\lambda L_i + L_j \to L_j} B$ with $i \neq j$, $\det B = \det A$.

   (b) The integers 2014, 1484, 3710 and 6996 are all multiples of 106. Show (without brute-force calculations) that also

$$\det \begin{pmatrix} 2 & 1 & 3 & 6 \\ 0 & 4 & 7 & 9 \\ 1 & 8 & 1 & 9 \\ 4 & 4 & 0 & 6 \end{pmatrix}$$

   is a multiple of 106.

Lösung:

(a) See lecture notes, Prop. 4.2.3.

(b) We add (1000 × (1. row) + 100 × (2. row) + 10 × (3. row)) to the 4th row. This does not change the determinant and we get

$$\det \begin{pmatrix} 2 & 1 & 3 & 6 \\ 0 & 4 & 7 & 9 \\ 1 & 8 & 1 & 9 \\ 4 & 4 & 0 & 6 \end{pmatrix} = \det \begin{pmatrix} 2 & 1 & 3 & 6 \\ 0 & 4 & 7 & 9 \\ 1 & 8 & 1 & 9 \\ 2014 & 1484 & 3710 & 6996 \end{pmatrix}.$$

As every entry of the last row is divisible by 106, we get

$$\det \begin{pmatrix} 2 & 1 & 3 & 6 \\ 0 & 4 & 7 & 9 \\ 1 & 8 & 1 & 9 \\ 2014 & 1484 & 3710 & 6996 \end{pmatrix} = 106 \cdot \det \begin{pmatrix} 2 & 1 & 3 & 6 \\ 0 & 4 & 7 & 9 \\ 1 & 8 & 1 & 9 \\ a_1 & a_2 & a_3 & a_4 \end{pmatrix}$$

with integers $a_1, \ldots, a_4$. As the determinant of a matrix over $\mathbb{Z}$ lies in $\mathbb{Z}$, the claim follows.

4. Compute the determinants of the following matrices

$$A = \begin{pmatrix} 0 & 0 & 0 & 1 \\ -1 & 2 & 0 & 1 \\ 1 & 2 & -3 & 1 \\ 0 & -4 & 2 & -1 \end{pmatrix}, \quad B = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 \\ -1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & -1 & 0 \\ 0 & 0 & -1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 \end{pmatrix},$$

$$C = \begin{pmatrix} 2 & -3 & 5 & 1 & 4 \\ 2 & -3 & 1 & -6 & 18 \\ 4 & -3 & 9 & 6 & 10 \\ -2 & 4 & -6 & -1 & -1 \\ -6 & 11 & -23 & -14 & 9 \end{pmatrix}.$$

Solution: Gaussian eliminiation yields

$$\begin{aligned} \det(A) &= -4 \\ \det(B) &= 0 \\ \det(C) &= 24. \end{aligned}$$

Alternatively one notices that the first column of B is a linear combination of the thrird and fifth column, from which $\det(B) = 0$ follows.

5. Let $A_n \in \mathrm{M}(n \times n, \mathbb{R})$ be the matrix

$$\begin{pmatrix} 2 & 1 & 0 & \dots & 0 \\ 1 & 2 & 1 & \dots & 0 \\ 0 & \ddots & \ddots & \ddots & \vdots \\ \vdots & & 1 & 2 & 1 \\ 0 & \dots & 0 & 1 & 2 \end{pmatrix}$$

Prove that

$$\det(A_n) = n + 1.$$

Solution: We use induction over n. For $n = 1, 2$, we obtain

$$\det(A_1) = 2, \det(A_2) = 4 - 1 = 3,$$

which agrees with our claim. Hence let $n > 2$ and assume that the claim is known for all $n' < n$. Laplace expansion after the first column yields

$$\det(A_n) = 2 \cdot \det(A_{n-1}) - \det(A_{n-2}) = 2n - (n-1) = n + 1.$$

Here we used the induction hypothesis, and for the second summand we expanded after the first row.

6. Let K be a subfield of the complex numbers and n a positive integer. Let $j_1, \dots, j_n$ and $k_1, \dots, k_n$ be positive integers not exceeding n. For an $n \times n$ matrix A over K define

$$D(A) = A(j_1, k_1)A(j_2, k_2) \cdots A(j_n, k_n).$$

Prove that D is n-linear if and only if the integers $j_1, \dots, j_n$ are distinct.

Lösung: We first assume that $j_r = j_s$ for some $r \neq s$ and show that D is not n-linear in this case. Let B be be the matrix obtained by multiplying the j_r-th row of A by $\lambda \in K$. Then,

$$\begin{aligned} D(B) &= \prod_{\substack{i=1 \\ j_i \neq j_r}}^{n} A(j_i, k_i) \cdot \prod_{\substack{k=1 \\ j_k = j_r}}^{n} (\lambda A(j_r, k_r)) \\ &= \lambda^m D(A), \end{aligned}$$

where $m \geqslant 2$ is the number of indices s for which $j_s = j_r$. Since in general $\lambda^m D(A) \neq \lambda D(A)$, e.g. for $\lambda = 2$ and non-zero matrix entries, this shows that D is not n-linear.

Note. In the case of $K := \mathbb{F}_2$, the field with 2 elements, the scalars are the additive identity 0_K and the multiplicative identity 1_K. It follows from the field axioms (check it!) that $\forall m \geqslant 1 : 0_K^m = 0_K$, and $\forall m \geqslant 0 : 1_K^m = 1_K$. Hence such a map D would be n-linear in this case. However, since $\mathbb{F}_2$ is not a subfield of $\mathbb{C}$ (to see why, try to define an injective field homomorphism from $\mathbb{F}_2$ into $\mathbb{C}$), this cannot happen here.

Let us now assume that the j_i's are pairwise distinct. Let $\lambda \in K$, let $1 \leqslant r \leqslant n$, let $\{\alpha_i\}_{i=1}^n \subset K$, and consider B to be the matrix obtained by multiplying the j_r-th row of A by λ then adding to it the row-vector $(\alpha_i)_{i=1}^n$. We compute that

$$\begin{aligned} D(B) &= \prod_{\substack{i=1 \\ i \neq r}}^{n} A(j_i, k_i) \cdot (\lambda A(j_r, k_r) + \alpha_{k_r}) \\ &= \lambda D(A) + D(C), \end{aligned}$$

where C is the matrix obtained by replacing the j_r-th row of A by $(\alpha_i)_{i=1}^n$. This shows that D is n-linear.

Single Choice. In each exercise, exactly one answer is correct.

1. For which $x \in \mathbb{R}$ do we have $\det \begin{pmatrix} 1 & 1 & 1 \\ 1 & x & 1 \\ 1 & 1 & x \end{pmatrix} = 1$?

   (a) $x = -2$

   ✓(b) $x = 2$

   (c) $x = -1$

   (d) $x = 1$

   Solution: The determinant is $(x-1)^2$, hence (b) is correct.

2. Let K be a field and $\mathrm{Mat}_{n\times n}(K)$ the vector space of $n \times n$-Matrizen over K. Which assertion is wrong in general?

   (a) A marix $A \in \mathrm{Mat}_{n\times n}(K)$ over K is invertible if and only if $\det(A) \neq 0$.

   (b) The determinant of an upper triangular matrix only depends on its diagonal entries.

   ✓(c) For every $n \geqslant 0$ the determinant is a linear map $\mathrm{Mat}_{n\times n}(K) \to K$.

   (d) For every $n > 0$ the determinant map $Mat_{n\times n}(K) \to K$ is surjektive.

   Solution: The determinant $\det(A)$ is linear in every column or resp. row if the remaining rows or resp. columns are fixed, but not linear in A itself. For example, we have $\det(\lambda A) = \lambda^n \det(A)$, which is in general not equal to $\lambda \det(A)$ ist. Hence (c) is wrong.

   We proved (a) and (b) in the lecture. Moreover, assertion (d) is correct, as the matrix obtained from the identity by changing the upper left entry by λ has determinant λ.

3. In general, which operation changes the determinant?

   ✓(a) Exchanging two rows.

   (b) Adding the multiple of one row to another

   (c) Transpose.

   (d) Replacement by similar matrix.

   Solution: Exchanging two rows leads to a change of sign of the determinant.

Multiple Choice Fragen.

1. Which of the following assertions are correct for arbitrary $A, B \in M_{n\times n}(\mathbb{R})$ with $n \geqslant 2$?

✓(a) We have $\det(AB) = \det(A)\det(B)$.

✓(b) From $\det(A) \neq 0$ it follows that the column vectors $a_1, \cdots, a_n$ of A are linearly independent.

✓(c) Es gilt $\det(AB) = \det(BA)$.

(d) For every non-zero real number λ we have $\det(\lambda A) = \lambda \det(A)$.

(e) We have $\det(A+B) = \det(A) + \det(B)$.

2. Let $\det\begin{pmatrix} a & b \\ c & d \end{pmatrix} = 4$. Which of the following statements are true?

(a) $\det\begin{pmatrix} 2a & 2b \\ 2c & 2d \end{pmatrix} = 8$.

✓(b) $\det\begin{pmatrix} a & b \\ c-a & d-b \end{pmatrix} = 4$.

✓(c) $\det\begin{pmatrix} a & b \\ c+2a & d+2b \end{pmatrix} = 4$.

✓(d) $\det\begin{pmatrix} a & b \\ 3c & 3d \end{pmatrix} = 12$.

Solution: Assertion (a) is wrong, because we have

$$\det\begin{pmatrix} 2a & 2b \\ 2c & 2d \end{pmatrix} = 2^2 \det\begin{pmatrix} a & b \\ c & d \end{pmatrix} = 2^2 \cdot 4 = 16.$$

Assertion (b) is correct, because we have

$$\det\begin{pmatrix} a & b \\ c-a & d-b \end{pmatrix} = \det\begin{pmatrix} a & b \\ c & d \end{pmatrix} - \det\begin{pmatrix} a & b \\ a & b \end{pmatrix} = 4.$$

Assertion (c) is correct, because we have

$$\det\begin{pmatrix} a & b \\ c+2a & d+2b \end{pmatrix} = \det\begin{pmatrix} a & b \\ c & d \end{pmatrix} + 2\det\begin{pmatrix} a & b \\ a & b \end{pmatrix} = 4.$$

Assertion (d) is correct. This follows from the multilinearity of the determinant:

$$\det\begin{pmatrix} a & b \\ 3c & 3d \end{pmatrix} = 3\det\begin{pmatrix} a & b \\ c & d \end{pmatrix} = 12.$$