D-MATH | Lineare Algebra I | HS 2022

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Musterlösung Serie 15

Determinant

1. Let K be a commutative ring with identity. If A is a 2×2 matrix over K, the classical adjoint of A is the 2×2 matrix adj A defined by

$$\operatorname{adj} A = \begin{pmatrix} A_{22} & -A_{12} \\ -A_{21} & A_{11} \end{pmatrix}$$

If det denotes the unique determinant function on 2×2 matrices over K, show that

(a) $(\operatorname{adj} A)A = A(\operatorname{adj} A) = (\det A)I$;

(b) $\det(\operatorname{adj} A) = \det(A)$;

(c) $\operatorname{adj}(A^t) = (\operatorname{adj} A)^t$.

(A^t denotes the transpose of A.)

Lösung: Each of the above equations follows from straightforward computations.

2. (a) List explicitly the 24 permutations of degree 4, state which are odd and which are even, and use this to give the complete Leibniz formula

$$\det(A) = \sum_{\sigma} (\operatorname{sgn} \sigma) A(1, \sigma 1) \cdots A(n, \sigma n)$$

for the determinant of a 4×4 matrix. Notice that for $n \geqslant 4$ it is not sufficient to compute a combination of the diagonals of a matrix to obtain its determinant.

(b) For a general $n \in \mathbb{N}_{\geqslant 1}$, how many *even* permutations are there in S_n?

Lösung:

Notation. We write $(a_1 a_2 \cdots a_r)$ for the cycle of length r that maps a_1 to a_2, a_2 to a_3, ..., a_{r-1} to a_r, and a_r to a_1. If σ_1, σ_2 are two cycles, we write their composition $\sigma_2 \sigma_1$.

(a) We count how many permutations of each type we should have. At the same time, we identify whether they are even or odd by counting the number of inversions (or crossings) as explained in the Abschnitt 4.3 of the lecture notes in German. Below, odd permutations are denoted with an asterisk.

- There are $\binom{4}{2} = 6$ transpositions: $(12)^*, (13)^*, (14)^*, (23)^*, (24)^*, (34)^*$;

- there are $\frac{4\cdot3\cdot2}{3} = 8$ 3-cycles: $(123), (213), (124), (214), (234), (324), (134), (314)$;
- there are $\frac{4!}{4} = 6$ 4-cycles: $(1234)^*, (1342)^*, (1243)^*, (1324)^*, (1432)^*, (1423)^*$;
- there are $\frac{6}{2} = 3$ products of 2 transpositions that are neither 3-cycles nor 4-cycles: $(12)(34), (13)(24), (14)(23)$. Note that neither of these can be 3-cycles nor 4-cycles since they are of order 2.
- Finally, there is the identity.

The Leibniz formula yields

$$\begin{aligned}\det A = &- a_{12}a_{21}a_{33}a_{44} - a_{13}a_{22}a_{31}a_{44} - a_{14}a_{22}a_{33}a_{41} - a_{11}a_{23}a_{32}a_{44}\\ &- a_{11}a_{24}a_{33}a_{42} - a_{11}a_{22}a_{34}a_{43} + a_{12}a_{23}a_{31}a_{44} + a_{13}a_{21}a_{32}a_{44}\\ &+ a_{12}a_{24}a_{33}a_{41} + a_{14}a_{21}a_{33}a_{42} + a_{11}a_{23}a_{34}a_{42} + a_{11}a_{24}a_{32}a_{43}\\ &+ a_{13}a_{22}a_{34}a_{41} + a_{14}a_{22}a_{31}a_{43} - a_{12}a_{23}a_{34}a_{41} - a_{13}a_{21}a_{34}a_{42}\\ &- a_{12}a_{24}a_{31}a_{43} - a_{13}a_{24}a_{32}a_{41} - a_{14}a_{21}a_{32}a_{43} - a_{14}a_{23}a_{31}a_{42}\\ &+ a_{12}a_{21}a_{34}a_{43} + a_{13}a_{24}a_{31}a_{42} + a_{14}a_{23}a_{32}a_{41} + a_{11}a_{22}a_{33}a_{44}.\end{aligned}$$

(b) First note that when $n = 1$, $S_1 = \{\mathrm{id}\}$. Hence the number of even permutations is 1.

For $n \geqslant 2$, we prove that S_n has as many even permutations as it has odd permutations by building a bijection between even and odd permutations. Let $\sigma \in S_n$ and consider $(\sigma(1)\,\sigma(2)) \circ \sigma$. Then

$$((\sigma(1)\,\sigma(2)) \circ \sigma)(i) = \begin{cases} \sigma(i), & i \in \{3, \dots, n\} \smallsetminus \{1, 2\} \\ \sigma(1), & i = 2 \\ \sigma(2), & i = 1 \end{cases}$$

Note that

$$\mathrm{sgn}((\sigma(1)\,\sigma(2)) \circ \sigma) = \mathrm{sgn}((\sigma(1)\,\sigma(2))) \cdot \mathrm{sgn}(\sigma) = -\,\mathrm{sgn}(\sigma).$$

Hence, multiplying by $(\sigma(1)\,\sigma(2))$ defines a map

$$\begin{array}{rccc}\varphi : & \{\text{even permutations of } S_n\} & \to & \{\text{odd permutations of } S_n\} \\ & \sigma & \mapsto & (\sigma(1)\,\sigma(2)) \circ \sigma\end{array}$$

This map is its own both-sided inverse and therefore is bijective, which shows that S_n splits equally between even and odd permutations and therefore that there are $n!/2$ of each.

Aliter: We use induction to prove that S_n contains $\frac{n!}{2}$ even permutations for $n \geqslant 2$. First notice that S_2 contains $1 = \frac{|S_2|}{2}$ even permutation (the identity) and $1 = \frac{|S_2|}{2}$ odd permutation (the only non-trivial one). Recall that you have looked at S_3 in the lectures and at S_4 above and that they also equally split in even and odd permutations.

Assume now that $n > 2$. Let $\sigma \in S_n$ and denote a_i the image $\sigma(i)$ for $i \in \{1, \ldots, n\}$. Then

$$((a_n n) \circ \sigma)(i) = \begin{cases} a_i, & i \neq n \wedge a_i \neq n \\ n, & i = n \\ a_n, & a_i = n \end{cases}$$

Hence the permutation $(a_n n) \circ \sigma$ can be viewed as an element of S_{n-1} denoted $\tilde{\sigma}$ satisfying

$$\sigma = (a_n n) \circ \tilde{\sigma}.$$

This equality shows that $\tilde{\sigma}$ is uniquely determined by the choice of σ. Hence we can define a map

$$\begin{array}{rccl} \varphi : & S_n & \rightarrow & S_{n-1} \\ & \sigma & \mapsto & \varphi(\sigma), \quad \text{such that } (a_n\, n) \circ \sigma = \varphi(\sigma). \end{array}$$

Observe that this map is surjective. Indeed, we can view any $\tau \in S_{n-1}$ as an element of σ of S_n that maps n to n. Then

$$\tau = (n\, n) \circ \sigma \implies \tau = \varphi(\sigma).$$

Additionally, observe that this map is n-to-1 since for any $\tau \in S_{n-1}$, for any $a_n \in \{1, \ldots, n\}$, the permutation $(a_n\, n)\tau \in S_n$ is a preimage of τ.

Finally, note that any even permutation $\tau \in S_{n-1}$ will have $n-1$ *odd* preimages and 1 *even* preimage (which corresponds to τ viewed as an element of S_n) and any odd permutation $\tau \in S_{n-1}$ will have $n-1$ *even* preimages and 1 *odd* preimage. Hence, using our induction hypothesis, the number of even permutations in S_n is

$$\frac{(n-1)!}{2} \cdot 1 + \frac{(n-1)!}{2} \cdot (n-1) = \frac{n!}{2}.$$

Aliter: We have showed that $\operatorname{sgn} : S_n \rightarrow \{-1, 1\}$ is a group homomorphism (the set on the right defines group under multiplication). By the first isomorphism theorem, we have

$$S_n \big/ \ker(\operatorname{sgn}) \cong \{-1, 1\}.$$

Since every group involved is finite, their cardinalities also match and

$$2 = \left| S_n \big/ \ker(\operatorname{sgn}) \right| = \frac{|S_n|}{|\ker(\operatorname{sgn})|} = \frac{n!}{|\ker(\operatorname{sgn})|}.$$

By definition, $\ker(\operatorname{sgn})$ is the group of even permutations. Hence there are $\frac{n!}{2}$ even permutations and thereupon an equal number of odd ones.

3. An $n \times n$ matrix A is called triangular if $A_{ij} = 0$ whenever $i > j$ or if $A_{ij} = 0$ whenever $i < j$. Prove that the determinant of a triangular matrix is the product $A_{11}A_{22} \cdots A_{nn}$ of its diagonal entries.

   Lösung: Assume that A is upper triangular and consider the Leibniz formula

$$\det A = \sum_{\sigma \in S_n} \operatorname{sgn}(\sigma) \prod_{i=1}^{n} A(i, \sigma(i)).$$

   A is upper triangular, so if for a given $\sigma \in S_n$ $\exists i \in \{1, \ldots, n\}$ s.t. $\sigma(i) < i$, the above product vanishes. Let us therefore only consider permutations σ such that $\forall i \in \{1, \ldots, n\} : \sigma(i) \geqslant i$.

   For such a permutation, assume that there exists an index $i \in \{2, \ldots, n\}$ such that $\sigma(i) > i$, and let i_0 be the smallest such index. Then σ maps $\{i_0, \ldots, n\}$ to $\{i_0 + 1, \ldots n\}$. This is a contradiction since σ is a bijective map. We deduce that

$$\begin{aligned} \det A &= \operatorname{sgn}(\mathrm{id}) \prod_{i=1}^{n} A(i, i) \\ &= \prod_{i=1}^{n} A(i, i). \end{aligned}$$

   Aliter: We prove it by induction on the size of the matrix n for upper triangular matrices. For $n = 1$, the equality clearly holds. For $n = 2$, we have

$$\det \begin{pmatrix} A(1,1) & A(1,2) \\ 0 & A(2,2) \end{pmatrix} = A(1,1)A(2,2).$$

   Assume that the formula holds for every upper triangular matrix of size $n - 1$. Consider

$$A = \begin{pmatrix} A(1,1) & A(2,2) & \cdots & A(1,n) \\ 0 & A(2,2) & \cdots & A(2,n) \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & A(n,n) \end{pmatrix}$$

   We compute $\det A$ by expanding with respect to the first row and use our induction hypothesis:

$$\begin{aligned} \det A &= (-1)^2 A(1,1) \cdot \det \begin{pmatrix} A(2,2) & A(2,3) & \cdots & A(2,n) \\ 0 & A(3,3) & \cdots & A(3,n) \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & A(n,n) \end{pmatrix} \\ &= \prod_{i=1}^{n} A(i,i). \end{aligned}$$

4. Let $n \in \mathbb{N}_{\geqslant 2}$. Show that

$$\det \begin{pmatrix} 1 & x_1 & x_1^2 & \cdots & x_1^{n-1} \\ 1 & x_2 & x_2^2 & \cdots & x_2^{n-1} \\ \vdots & \vdots & \vdots & & \vdots \\ 1 & x_n & x_n^2 & \cdots & x_n^{n-1} \end{pmatrix} = \prod_{1 \leqslant i < j \leqslant n} (x_j - x_i).$$

Remark. Products of the sort are called a *Vandermonde determinants* and the above matrix is called a *Vandermonde matrix.*

Lösung: We prove the formula by induction. First note that the formula holds for $n = 2$ since

$$\det \begin{pmatrix} 1 & x_1 \\ 1 & x_2 \end{pmatrix} = x_2 - x_1.$$

Let us now consider the n-th Vandermonde matrix, denoted V_n, and assume that the formula holds for the $n-1$-st determinant. We substract the first row from every other row (this operation preserved the determinant), then expand with respect to the first column and obtain

$$\begin{aligned} \det V_n &= \det \begin{pmatrix} 1 & x_1 & \cdots & x_1^{n-1} \\ 0 & x_2 - x_1 & \cdots & x_2^{n-1} - x_1^{n-1} \\ \vdots & \vdots & & \vdots \\ 0 & x_n - x_1 & \cdots & x_n^{n-1} - x_1^{n-1} \end{pmatrix} \\ &= 1 \cdot \det \begin{pmatrix} x_2 - x_1 & \cdots & x_2^{n-1} - x_1^{n-1} \\ \vdots & & \vdots \\ x_n - x_1 & \cdots & x_n^{n-1} - x_1^{n-1} \end{pmatrix}. \end{aligned}$$

Recall the formula

$$x^m - y^m = (x - y)(x^{m-1} + x^{m-2}y + \cdots + xy^{m-2} + y^{m-1}).$$

For each $i \in \{2, \ldots, n\}$, we factor the $i-1$-st row by $(x_i - x_1)$ and we use the n-linearity of the determinant to pull this factor out. We obtain

$$\det V_n = \prod_{i=2}^{n} (x_i - x_1) \det \begin{pmatrix} 1 & x_2 + x_1 & \cdots & \sum_{k=0}^{n-2} x_1^k x_2^{n-2-k} \\ \vdots & \vdots & & \vdots \\ 1 & x_n + x_1 & \cdots & \sum_{k=0}^{n-2} x_1^k x_n^{n-2-k} \end{pmatrix}.$$

Note that

$$\begin{pmatrix} 1 & x_i & x_i^2 & \cdots & x_i^{n-2} \end{pmatrix} \cdot \begin{pmatrix} x_1^m \\ x_1^{m-1} \\ \vdots \\ 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix} = \sum_{k=0}^{m} x_1^k x_i^{m-k}.$$

Hence the last matrix can be written as the product of the Vandermonde matrix

$$\begin{pmatrix} 1 & x_2 & \cdots & x_2^{n-2} \\ \vdots & \vdots & & \vdots \\ 1 & x_n & \cdots & x_n^{n-2} \end{pmatrix}$$

with the upper triangular matrix

$$\begin{pmatrix} 1 & x_1 & x_1^2 & \cdots & x_1^{n-2} \\ 0 & 1 & x_1 & \cdots & x_1^{n-3} \\ \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & 0 & \cdots & 1 \end{pmatrix}.$$

We use our induction hypothesis and the fact that the determinant of a product of square matrices equals the product of the determinants, and conclude that

$$\begin{aligned} \det V_n &= \prod_{i=2}^{n}(x_i - x_1) \cdot \prod_{2 \leqslant j < k \leqslant n-1} (x_k - x_j) \cdot 1 \\ &= \prod_{1 \leqslant i < j \leqslant n} x_j - x_i. \end{aligned}$$

5. Let K be a field and let $A, B, C, D \in M_{n \times n}(K)$. Assume that A and C commute and that $\det A \neq 0$. Show that

$$\det \left(\begin{array}{c|c} A & B \\ \hline C & D \end{array} \right) = \det(A \cdot D - C \cdot B)$$

Hint. Consider the matrix

$$\left(\begin{array}{c|c} I_n & O_n \\ \hline -C & A \end{array} \right).$$

Lösung: We compute that

$$\begin{aligned} \left(\begin{array}{c|c} I_n & O_n \\ \hline -C & A \end{array} \right) \cdot \left(\begin{array}{c|c} A & B \\ \hline C & D \end{array} \right) &= \left(\begin{array}{c|c} A & B \\ \hline -C \cdot A + A \cdot C & -C \cdot B + A \cdot D \end{array} \right) \\ &= \left(\begin{array}{c|c} A & B \\ \hline O_n & -C \cdot B + A \cdot D \end{array} \right), \end{aligned}$$

where we used the fact that A and C commute to obtain the last equality. Using the formula for the determinant of the product of two matrices and a result you've seen in the lectures about block matrices of this shape, we obtain

$$\begin{aligned} & \det(I_n) \det(A) \det \left(\begin{array}{c|c} A & B \\ \hline C & D \end{array} \right) = \det(A) \det(A \cdot D - C \cdot B) \\ \iff \quad & \det \left(\begin{array}{c|c} A & B \\ \hline C & D \end{array} \right) = \det(A \cdot D - C \cdot B). \end{aligned}$$

6. Prove the following proposition using the Leibniz formula for determinants:

Proposition. *Let K be a field, and let $A, B \in M_{n\times n}(K)$. Then*

$$\det(AB) = \det(A)\cdot\det(B).$$

Hint. Denote $\{\mathbf{e}_i\}_{i=1}^n$ the standard basis of K^n and write the matrix B as a list of column blocks:

$$B = \left(\sum_{s_1=1}^n B(s_1,1)\mathbf{e}_{s_1} \;\middle|\; \cdots \;\middle|\; \sum_{s_n=1}^n B(s_n,n)\mathbf{e}_{s_n} \right).$$

You will also need to prove the following lemma

Lemma. For any $A \in M_{n\times n}(K)$ and any $\sigma \in S_n$, we have

$$\det\left(A\cdot\mathbf{e}_{\sigma(1)} \;\middle|\; \cdots \;\middle|\; A\cdot\mathbf{e}_{\sigma(n)} \right) = \operatorname{sgn}(\sigma)\det(A).$$

Lösung: We write the matrix B as a list of column blocks:

$$B = \left(\sum_{s_1=1}^n B(s_1,1)\mathbf{e}_{s_1} \;\middle|\; \cdots \;\middle|\; \sum_{s_n=1}^n B(s_n,n)\mathbf{e}_{s_n} \right).$$

We can now write the product $A\cdot B$ as list of column blocks:

$$A\cdot B = \left(\sum_{s_1=1}^n B(s_1,1)A\cdot\mathbf{e}_{s_1} \;\middle|\; \cdots \;\middle|\; \sum_{s_n=1}^n B(s_n,n)A\cdot\mathbf{e}_{s_n} \right).$$

To compute the determinant, we expand with respect to each column:

$$\det(AB) = \sum_{s_1=1}^n \cdots \sum_{s_n=1}^n \left(\prod_{i=1}^n B(s_i,i)\right)\det\left(A\cdot\mathbf{e}_{s_1} \;\middle|\; \cdots \;\middle|\; A\cdot\mathbf{e}_{s_n} \right)$$

Note that if for some $i \neq j : s_i = s_j$, the s_i-th column of A appears several times in the last matrix. Hence the determinant vanishes in this case. So we only need to consider tuples $(s_1,\dots,s_n)$ for which $i \mapsto s_i$ is a bijection. Such tuples correspond 1-to-1 to the permutations S_n, therefore we can rewrite the last equation as

$$\sum_{\sigma\in S_n}\left(\prod_{i=1}^n B(\sigma(i),i)\right)\det\left(A\cdot\mathbf{e}_{\sigma(1)} \;\middle|\; \cdots \;\middle|\; A\cdot\mathbf{e}_{\sigma(n)} \right). \tag{1}$$

We'll use the following result to conclude:

Lemma 1. *For any $A \in M_{n\times n}(K)$ and any $\sigma \in S_n$, we have*

$$\det\left(A\cdot\mathbf{e}_{\sigma(1)} \;\middle|\; \cdots \;\middle|\; A\cdot\mathbf{e}_{\sigma(n)} \right) = \operatorname{sgn}(\sigma)\det(A).$$

Beweis. You have seen that any permutation can be written as a product of transpositions. We prove the lemma by induction on the number r of transpositions used to decompose σ. Assume first that σ is a transposition. Then σ inverts 2 columns of A. Hence, as seen in the previous exercise sheet,

$$\det \left(\ A \cdot \mathbf{e}_{\sigma(1)} \ \middle| \ \cdots \ \middle| \ A \cdot \mathbf{e}_{\sigma(n)} \ \right) = -\det A.$$

Assume now that σ is a product of $r > 1$ transpositions, $\sigma = \tau_r \tau_{r-1} \ldots \tau_1$. Denote $\tau_{r-1} \cdots \tau_1$ by σ_1. We have

$$\det \left(\ A \cdot \mathbf{e}_{\sigma(1)} \ \middle| \ \cdots \ \middle| \ A \cdot \mathbf{e}_{\sigma(n)} \ \right) = \det \left(\ A \cdot \mathbf{e}_{\tau_r \sigma_1(1)} \ \middle| \ \cdots \ \middle| \ A \cdot \mathbf{e}_{\tau_r \sigma_1(n)} \ \right).$$

Since τ_r permutes two columns of $\left(\ A \cdot \mathbf{e}_{\sigma_1(1)} \ \middle| \ \cdots \ \middle| \ A \cdot \mathbf{e}_{\sigma_1(n)} \ \right)$, we have again that

$$\det \left(\ A \cdot \mathbf{e}_{\tau_r \sigma_1(1)} \ \middle| \ \cdots \ \middle| \ A \cdot \mathbf{e}_{\tau_r \sigma_1(n)} \ \right) = -\left(\ A \cdot \mathbf{e}_{\sigma_1(1)} \ \middle| \ \cdots \ \middle| \ A \cdot \mathbf{e}_{\sigma_1(n)} \ \right)$$

We conclude using the induction hypothesis. □

We plug the result we just obtained in (1) and obtained

$$\begin{aligned} \det(AB) &= \det(A) \sum_{\sigma \in S_n} \operatorname{sgn}(\sigma) \left(\prod_{i=1}^{n} B(\sigma(i), i) \right) \\ &= \det(A) \det(B). \end{aligned}$$