D-MATH | Lineare Algebra II | FS 2023

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CHARACTERISTIC POLYNOMIALS, EIGENVECTORS, EIGENVALUES

1. Consider the matrix $A = \begin{pmatrix} 3 & 0 & -2 \\ 2 & 0 & -2 \\ 0 & 1 & 1 \end{pmatrix}$ over $\mathbb{R}$.

   (a) Determine the characteristic polynomial of A.

   (b) Determine the eigenvalues of A.

   (c) The *geometric multiplicity* of an eigenvector is the dimension of its eigenspace. The *arithmetic multiplicity* of an eigenvector is the multiplicity of this eigenvector as a zero of the characteristic polynomial. Determine the arithmetic and geometric multiplicity of all eigenvalues.

Solution:

(a) We use the determinant formula for 3×3-matrices:

$$\begin{aligned} \operatorname{char}_A(X) = \det(X \cdot I_3 - A) = \det \begin{pmatrix} X-3 & 0 & 2 \\ -2 & X & 2 \\ 0 & -1 & X-1 \end{pmatrix} \\ = (X-3)X(X-1) + 4 + 2(X-3) \\ = X^3 - 4X^2 + 5X - 2. \end{aligned}$$

(b) As the polyonial is monic and has coefficients in $\mathbb{Z}$, all of its zeros in $\mathbb{Q}$ are contained in $\mathbb{Z}$ and divide the constant coefficient -2. Trying out yields $X = 1$. Polynomial division and the quadratic formula yield

$$\operatorname{char}_A(X) = (X-1)\left(X^2 - 3X + 2\right) = (X-1)^2(X-2).$$

Thus the eigenvalues are $\lambda_1 := 1$ and $\lambda_2 := 2$.

(c) The arithmetic multiplicity is the multiplicity of the eigenvalues as zeros of the characteristic polynom. Thus the arithmetic multiplicity of λ_1 is 2 and that of λ_2 is 1. The geometric multiplicity is the dimension of the eigenspace. For the eigenvalue λ_1, consider the matrix

$$\lambda_1 \cdot I_3 - A = \begin{pmatrix} -2 & 0 & 2 \\ -2 & 1 & 2 \\ 0 & -1 & 0 \end{pmatrix}.$$

This matrix has rank 2, because the first two columns are linearly independent and the last column is minus the first one. The kernel of the corresponding linear map hence has dimension $3-2=1$. Therefore, the geometric multiplicity of λ_1 is 1 . Since the eigenvalue λ_2 has arithmetic multiplicity 1, the geometric multiplicity is $\leqslant 1$ and >0; thus, it is 1.

Aliter: Consider for λ_2 the matrix

$$\lambda_2 \cdot I_3 - A = \begin{pmatrix} -1 & 0 & 2 \\ -2 & 2 & 2 \\ 0 & -1 & 1 \end{pmatrix}.$$

This matrix has rank 2, and thus the geometric multiplicity of λ_2 is $3-2=1$.

2. Compute the characteristic polynomial, the eigenvalues and eigenvectors of the following matrices over $\mathbb{Q}$ and check if they are diagonalizable.

   (a) $A := \begin{pmatrix} 1 & -1 \\ 2 & 4 \end{pmatrix}$

   (b) $B := \begin{pmatrix} 2 & 2 & 3 \\ 1 & 2 & 1 \\ 2 & -2 & 1 \end{pmatrix}$

   (c) $C := \begin{pmatrix} -4 & -3 & -1 & -7 \\ -3 & -1 & -1 & -4 \\ 6 & 4 & 3 & 8 \\ 3 & 3 & 1 & 6 \end{pmatrix}$

   Solution:

   (a) The matrix A has characteristic polynomial

   $$\operatorname{char}_A(X) = X^2 - 5X + 6 = (X-2)(X-3)$$

   and hence eigenvalues 2 and 3, both with arithmetic multiplicity 1. The eigenspaces $E_{\lambda,A}$ corresponding to eigenvalue λ are

   $$E_{2,A} = \left\langle \begin{pmatrix} 1 \\ -1 \end{pmatrix} \right\rangle, \quad E_{3,A} = \left\langle \begin{pmatrix} 1 \\ -2 \end{pmatrix} \right\rangle.$$

   As for both eigenvalues of A the geometric multiplicity is the arithmetic multiplicity, the matrix A is diagonalizable.

   (b) The matrix B has the characteristic polynomial

   $$\operatorname{char}_B(X) = X^3 - 5X^2 + 2X + 8 = (X-4)(X-2)(X+1).$$

and thus its eigenvalues are $4, 2, -1$, all of arithmetic multiplicity 1. Their eigenspaces are

$$E_{4,B} = \left\langle \begin{pmatrix} 8 \\ 5 \\ 2 \end{pmatrix} \right\rangle, \quad E_{2,B} = \left\langle \begin{pmatrix} 2 \\ 3 \\ -2 \end{pmatrix} \right\rangle, \quad E_{-1,B} = \left\langle \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix} \right\rangle.$$

As for every eigenvector of B the geometric multiplicity and the geometric multiplicity ar ethe same, the matrix B is diagonalizable.

(c) The matrix C has characteristic polynomial

$$\operatorname{char}_C(X) = X^4 - 4X^3 + 3X^2 + 4X - 4 = (X-1)(X+1)(X-2)^2$$

and hence the eigenvalues $1, -1, 2$ with arithmetic multiplicities $1, 1, 2$. The eigenspaces are

$$E_{1,C} = \left\langle \begin{pmatrix} -1 \\ -1 \\ 1 \\ 1 \end{pmatrix} \right\rangle, \quad E_{-1,C} = \left\langle \begin{pmatrix} -2 \\ -1 \\ 2 \\ 1 \end{pmatrix} \right\rangle, \quad E_{2,C} = \left\langle \begin{pmatrix} -1 \\ -1 \\ 2 \\ 1 \end{pmatrix} \right\rangle.$$

The eigenvalue 2 has arithmetic multiplicity 2 and geometric multiplicity 1. Therefore, the matrix C is not diagonalizable.

3. For an arbitrary invertible $n \times n$-matrix A, write the characteristic polynomial of A^{-1} in terms of the characteristic polynomial of A.

Solution: We have

$$\begin{aligned} \operatorname{char}_{A^{-1}}(X) &= \det\left(X \cdot I_n - A^{-1}\right) \\ &= \det\left((-X) \cdot A^{-1} \cdot \left(X^{-1} \cdot I_n - A\right)\right) \\ &= (-X)^n \det\left(A^{-1}\right) \det\left(X^{-1} \cdot I_n - A\right) \\ &= \frac{(-X)^n}{\det(A)} \cdot \operatorname{char}_A\left(X^{-1}\right). \end{aligned}$$

4. Let K^∞ be the vectorspace of all infinite sequences in K, and let K_0^∞ be the subspace of all sequences where all but finitely many elements are 0.

(a) Determine all eigenvalues and eigenvectors of the endomorphism

$$T : K^\infty \to K^\infty,\ (x_0, x_1, x_2, \ldots) \mapsto (x_1, x_2, x_3, \ldots).$$

(b) Do the same for the induced endomorphism $K_0^\infty \to K_0^\infty$.

(c) Construct an endomorphism of K_0^∞ with eigenvalues $0, 1, 2, 3, \ldots$

(d) Construct an endomorphism of K_0^∞ which has no Eigenvalues.

Solution:

(a) Let $x = (x_i)_{i \geqslant 0} \in F$ and $\lambda \in K$ be such that

$$Tx = \lambda x$$

Then we have $\lambda x_n = x_{n+1}$ for all $n \geqslant 0$, which by induction yields $x_n = \lambda^n x_0$ for all n. The vector x is thus a multiple of

$$v_\lambda := \big(1, \lambda, \lambda^2, \ldots\big) \neq 0$$

On the other hand every non-zero multiple of v_λ is an eigenvector of T with eigenvalue λ. Hence every scalar $\lambda \in K$ is an eigenvalue of T with one-dimensional eigenspace $\langle v_\lambda \rangle$.

(b) Let

$$T_0 = T|_{F_0} : F_0 \to F_0$$

be the restriction of T on F_0. Because of (a), every eigenvector to the eigenvalue λ of T_0 is equal to cv_λ for $c \in K^\times$. But

$$cv_\lambda = (c, c\lambda, \ldots, c\lambda^n, \ldots)$$

is contained in F_0 if and only if $\lambda = 0$. Hence T_0 has eigenvalue 0 with eigenspace $\langle v_0 \rangle$ hat and no other eigenvalues.

(c) Consider the linear map

$$U : F_0 \to F_0, (x_n)_{n \geqslant 0} \mapsto (n \cdot x_n)_{n \geqslant 0}$$

For every $k \geqslant 0$ the vector $v_k := (\delta_{kn})_{n \geqslant 0}$ is an eigenvector of U with eigenvalue k. As the set $\{v_k \mid k \geqslant 0\}$ is a basis of F_0, there are up to scalars no other eigenvectors, hence no other eigenvalues.

(d) Consider the linear map

$$M : F_0 \to F_0, (x_0, x_1, x_2, \ldots) \mapsto (0, x_0, x_1, x_2, \ldots)$$

Let $x = (x_n)_{n \geqslant 0} \in F_0$ be a vector with $Mx = \lambda x$ for $\lambda \in K$. Then

$$x_0 = \lambda x_1 = \cdots = \lambda^n x_n = \ldots$$

As there exists an $m \geqslant 0$ with $x_k = 0$ for all $k \geqslant m$, we have $x_0 = x_1 = \cdots = x_m = 0$, hence $x = 0$. The endomorphism M therefore has no eigenvectors and thus no eigenvalues.

5. Let A be a nilpotent $n \times n$ real matrix. This means that there exists $m \geqslant 1$ with $A^m = O$. Show that the only possible eigenvalue of A is 0. When exactly is 0 an eigenvalue of A?

   Solution: Let $\lambda \in K$ be an eigenvalue of A with eigenvector v. Then we have $Av = \lambda v$, and by induction we get $A^k v = \lambda^k v$ for all $k \geqslant 0$. By assumption we have $\lambda^m v = A^m v = Ov = 0$. As $v \neq 0$, we get $\lambda = 0$. Hence $\lambda = 0$ is the only possible eigenvector A.

   We prove that 0 is always an eigenvalue of A. If A were invertible, the matrix A^m would be the product of invertible matrices, contradicting $A^m = O$. Hence A is not invertible. This implies that (the map "left multiplication by") A has non-trivial kernel and therefore that 0 is an eigenvalue of A.

   Aliter: For $n \geqslant 1$, we have $A^0 = I_n \neq O$. The smallest natural number $m \geqslant 1$ with $A^m = 0$ thus satisfies $A^{m-1} \neq 0$. Hence there exists a vector $v \in K^n$ with $w := A^{m-1}v \neq 0$. As
   $$Aw = A^m v = 0 \cdot v = 0$$
   we get that w is an eigenvector of A with eigenvalue 0.

6. Let V be a K-vectorspace and let $F, G \in \mathrm{End}(V)$. Show:

   (a) If $v \in V$ is an eigenvector of $F \circ G$ with eigenvalue λ and $G(v) \neq 0$, then $G(v)$ is an eigenvector of $G \circ F$ with eigenvalue λ.

   (b) If V is finite-dimensional, the endomorphisms $F \circ G$ and $G \circ F$ have the same eigenvalues.

   (c) Find a counterexample to (b) if V is not finite-dimensional.

   Solution:

   (a) Let $v \in V$ be an eigenvector of $F \circ G$ with eigenvalue λ with $G(v) \neq 0$. Then we have
   $$G \circ F(G(v)) = G(F \circ G(v)) = G(\lambda v) = \lambda G(v).$$
   Hence $G(v)$ is an eigenvector of $G \circ F$ with eigenvalue λ.

   (b) Let (λ, v) be an eigenvector-eigenvalue pair of $F \circ G$. We differentiate the cases $G(v) \neq 0$ and $G(v) = 0$.
   If $G(v) \neq 0$, then λ is an eigenvalue of $G \circ F$ by (a).
   If $G(v) = 0$, then $\lambda v = (F \circ G)(v) = F(0) = 0$. So, $\lambda = 0$ and we must show that 0 is an eigenvalue of $G \circ F$. This is equivalent to $G \circ F$ having non-trivial kernel, which happens if and only if $G \circ F$ is of rank $< \dim(V)$. Now,
   $$\mathrm{rank}(G \circ F) \leqslant \min(\mathrm{rank}(G), \mathrm{rank}(F)) < \dim(V)$$
   since G is an endomorphism of V with non–trivial kernel by assumption. Hence, 0 is an eigenvalue of $G \circ F$.

This shows that every eigenvalue of $F \circ G$ is an eigenvalue of $G \circ F$. The converse inclusion is obtained by exchanging G and F above.

(c) Let $V = \mathbb{R}^{\mathbb{N}} := \left\{(a_n)_{n \geqslant 0}\right\}$ be the vector space of all sequences in $\mathbb{R}$. Define linear maps $F, G : V \to V$ by

$$F : (a_0, a_1, a_2, \ldots) \mapsto (0, a_1, a_2, a_3, \ldots)$$
$$G : (a_0, a_1, a_2, \ldots) \mapsto (a_1, a_2, \ldots)\,.$$

Then $G \circ F$ is the identity with sole Eigenvalue 1, whereas $F \circ G$ has also eigenvalue 0 because

$$(F \circ G)(1, 0, 0, \cdots) = (0, 0, 0, \cdots).$$