D-MATH | Lineare Algebra II | FS 2023

Prof. M. Einsiedler
Prof. P. Biran

Musterlösung Serie 17

Eigenvectors, Eigenvalues

1. In each of the following cases, let T_i be the endomorphism of $\mathbb{R}^2$ which is represented by the matrix A_i in the standard ordered basis for $\mathbb{R}^2$, and let U_i be the endomorphism of $\mathbb{C}^2$ represented by A_i in the standard ordered basis. Find the characteristic polynomial for T_i and that for U_i, find the eigenvalues of each endomorphism, and for each such eigenvalue find a basis for the corresponding space of eigenvectors.

$$A_1 = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \quad A_2 = \begin{pmatrix} 2 & 3 \\ -1 & 1 \end{pmatrix}, \quad A_3 = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}.$$

Solution: We have

$$\mathrm{char}_{T_1}(X) = \mathrm{char}_{U_1}(X) = X(X-1).$$

Hence both T_1 and U_1 have real eigenvalues 0 and 1. We compute that

$$\mathrm{Eig}_{T_1}(0) = \left\langle \begin{pmatrix} 0 \\ 1 \end{pmatrix} \right\rangle_{\mathbb{R}}, \quad \mathrm{Eig}_{T_1}(1) = \left\langle \begin{pmatrix} 1 \\ 0 \end{pmatrix} \right\rangle_{\mathbb{R}}$$

and similarly for U_1 but taking the span over $\mathbb{C}$ instead of $\mathbb{R}$.

In the second case we have

$$\mathrm{char}_{T_2}(X) = X^2 - 3X + 5 \in \mathbb{R}[X].$$

This polynomial doesn't split into linear factors in $\mathbb{R}[X]$ hence the endomorphism T_2 does not have any real eigenvalues. However, if we now consider $U_2 \in \mathrm{End}(\mathbb{C}^2)$, $\mathrm{char}_{U_2}(X) \in \mathbb{C}[X]$ splits into linear factors and takes $\lambda_1 = \frac{1}{2}(3 + i\sqrt{11})$ and $\lambda_2 = \frac{1}{2}(3 - i\sqrt{11})$ for roots.

We use a handy trick to easily compute eigenvectors:

Eigenvector trick for 2×2 matrices. *Let A be a 2×2 matrix, and let λ be a (real or complex) eigenvalue of A. Then*

$$A - \lambda I_2 = \begin{pmatrix} \alpha & \beta \\ * & * \end{pmatrix} \implies \begin{pmatrix} -\beta \\ \alpha \end{pmatrix} \text{ is an eigenvector with eigenvalue } \lambda,$$

assuming the first row of $A - \lambda I_2$ is non-zero.

Explanation. Indeed since λ is an eigenvalue, $A - \lambda I_2$ has nontrivial kernel. It follows that its rows are collinear, i.e. that the second row is a complex multiple of the first one:

$$\begin{pmatrix} \alpha & \beta \\ * & * \end{pmatrix} = \begin{pmatrix} \alpha & \beta \\ \gamma\alpha & \gamma\beta \end{pmatrix}, \quad \text{for some } \gamma \in \mathbb{C}.$$

So $\left(\begin{smallmatrix} -\beta \\ \alpha \end{smallmatrix}\right)$ is an obvious element of that kernel.

Applying this to A_2 and its eigenvalue λ_1, we get

$$A_2 - \lambda_1 I_2 = \begin{pmatrix} 2-\lambda_1 & 3 \\ * & * \end{pmatrix}$$

which implies that $\mathrm{Eig}_{U_2}(\lambda_1)$ is generated by

$$\begin{pmatrix} -3 \\ 2-\lambda_1 \end{pmatrix} = \begin{pmatrix} -3 \\ \frac{1}{2}(1 - i\sqrt{11}) \end{pmatrix}.$$

Similarly, we recover that $\mathrm{Eig}_{U_2}(\lambda_2)$ is generated by

$$\begin{pmatrix} -3 \\ \frac{1}{2}(1 + i\sqrt{11}). \end{pmatrix}$$

We treat T_3 and U_3 similarly as T_1 and U_1. Their characteristic polynomial splits into linear factors in $\mathbb{R}[X]$:

$$\mathrm{char}_{T_3}(X) = \mathrm{char}_{U_3}(X) = X(X-2).$$

The eigenvalues are therefore 0 and 2. We find that

$$\mathrm{Eig}_{T_3}(0) = \left\langle \begin{pmatrix} -1 \\ 1 \end{pmatrix} \right\rangle_{\mathbb{R}}, \quad \mathrm{Eig}_{T_3}(2) = \left\langle \begin{pmatrix} 1 \\ 1 \end{pmatrix} \right\rangle_{\mathbb{R}}.$$

Similarly for U_3 over $\mathbb{C}$.

2. Let K be a field and let V be a finite-dimensional vector space over K. Suppose that $T \in \mathrm{End}(V)$ is invertible. Prove that $\mathrm{Eig}_T(\lambda) = \mathrm{Eig}_{T^{-1}}(1/\lambda)$ for every $\lambda \in K^*$.

   Lösung: This easily follows from the definition:

$$\begin{aligned} \mathrm{Eig}_T(\lambda) &= \{v \in V \mid Tv = \lambda v\} \\ &= \{w \in V \mid w = \lambda T^{-1} w\} \\ &= \left\{ w \in V \mid \frac{1}{\lambda} w = T^{-1} w \right\} \\ &= \mathrm{Eig}_{T^{-1}}\left(\frac{1}{\lambda}\right). \end{aligned}$$

Remark. We didn't use the assumption that V is finite-dimensional. In fact, this statement also holds in infinite-dimensional vector spaces by the same proof.

3. Consider the space $C^\infty(\mathbb{R})$ of smooth functions over $\mathbb{R}$ and the map

$$T: \begin{array}{ccc} C^\infty(\mathbb{R}) & \to & C^\infty(\mathbb{R}) \\ f & \mapsto & f' \end{array}$$

Find the eigenvalues and the corresponding eigenfunctions (this is a synonym for eigenvectors when working on a space whose elements are functions) of T.

Solution: For $\lambda \in K$, we can get an idea of the solution by solving the linear ordinary differential equation

$$\frac{\mathrm{d}\, f(x)}{\mathrm{d}\, x} = \lambda f(x).$$

We have

$$\begin{aligned} & \frac{1}{f(x)} \frac{\mathrm{d}\, f(x)}{\mathrm{d}\, x} = \lambda \\ \implies & \int \frac{1}{f(x)} \frac{\mathrm{d}\, f(x)}{\mathrm{d}\, x} \mathrm{d}\, x = \int \lambda \,\mathrm{d}\, x \end{aligned}$$

Substituting u for $f(x)$ on the left-hand side, we obtain $\mathrm{d}\, u = \frac{\mathrm{d}\, f(x)}{\mathrm{d}\, x} \mathrm{d}\, x$

$$\begin{aligned} & \int \frac{1}{u} \mathrm{d}\, u = \lambda x + C, \quad C \in K \\ \implies & \log(u) = \lambda x + C \\ \implies & \log(f(x)) = \lambda x + C \\ \implies & f(x) = e^{\lambda x + C}. \end{aligned}$$

Hence, the family $\{f(x) = f(0)e^{\lambda x} \mid \lambda \in K\}$ is a set of eigenfunctions of T and, at least formally, eigenfunctions should have the form $f(x) = f(0)e^{\lambda x}$ for $\lambda \in K$.

To check that these are the only possible solutions, we use the following trick: consider a solution f_0 of the differential equation $\frac{\mathrm{d}\, f(x)}{\mathrm{d}\, x} = \lambda f(x)$ and define the modified function

$$g_0(x) = e^{-\lambda x} f_0(x).$$

Now,

$$\begin{aligned} \frac{\mathrm{d}\, g_0}{\mathrm{d}\, x}(x) &= -\lambda e^{-\lambda x} f_0(x) + \lambda e^{-\lambda x} f_0(x) \\ &= 0. \end{aligned}$$

We deduce that $g_0(x)$ is constant, hence for all $x \in \mathbb{R}$,

$$g_0(x) = g_0(0) = f_0(0) \Leftrightarrow f_0(x) = f_0(0)e^{\lambda x}.$$

4. Let $K = \mathbb{R}$, show that K^∞ does not admit any countable basis.

   Hint: Use the fact that pairwise distinct eigenvalues correspond to a set of linearly independent eigenvectors.

   Solution: Let $\lambda \in K$ and consider the sequence $L_\lambda := (1, \lambda, \lambda^2, \lambda^3, \dots)$. Apply the shift operator

$$S : \begin{array}{ccc} K^\infty & \to & K^\infty \\ (a_0, a_1, a_2, a_3, \dots) & \mapsto & (a_1, a_2, a_3, \dots) \end{array}$$

   to L_λ and observe that (λ, L_λ) is an eigenvalue-eigenvector pair for S. Hence S admits an uncountable number of eigenvalues since each $\lambda \in K$ is one. Moreover, as seen in the lectures, since these eigenvalues are distinct, the set $\{L_\lambda \mid \lambda \in K\}$ is linearly independent. Using Serie 6 exercise 5, we conclude that K^∞ does not admit any countable basis.

5. (a) Let f be an endomorphism of a finite-dimensional vector space V, and let $V = V_1 \oplus \dots \oplus V_r$ with f-invariant subpaces V_i. Show, that the arithmetic, resp. geometric multiplicities of an eigenvalue $\lambda \in K$ of f is equal to the sum of the arithmetic, resp. geometric multiplies of λ as an eigenvalue of the endomorphisms $f|_{V_i}$ of V_i.

   (b) Deduce that f is diagonalizable if and only if $f|_{V_i}$ is diagonalizable for every i.

   (c) Let f and g be endomorphisms for the same finite dimensional vector space V. Show that f and g are *simultaneously diagonalizable* (meaning that there exists a basis of eigenvectors of f which are all also eigenvectors of g) if and only if they commute and are diagonalizable.

   Hint: To prove the the backward direction, first show that each eigenspace of f is g-invariant, i.e. that g maps eigenvectors of f to eigenvectors of f *in the same eigenspace.*

   Lösung:

   (a) For every $1 \leqslant i \leqslant r$ choose an ordered basis B_i of V_i. Joined in ascending order, these form a basis B of V. The transformation matrix of f with respect to B is then a block diagonal matrix with diagonal blocks $M_{B_i}^{B_i}(f|_{V_i})$ for $1 \leqslant i \leqslant r$. The characteristic polynomial of f thus is the product of the characteristic polynomials of $f|_{V_i}$; i.e.

$$\mathrm{char}_f(X) = \prod_{i=1}^{r} \mathrm{char}_{f|_{V_i}}(X) \tag{1}$$

   For every $\lambda \in K$ the arithmetic multiplicity of λ as eigenvalue of f is therefore equal to the sum over $1 \leqslant i \leqslant r$ of the arithmetic multiplicity of λ as eigenvalue of $f|_{V_i}$.

Now consider an arbitrary vector $v = v_1 + \cdots + v_r$ with $v_i \in V_i$. Then we have $f(v) = f(v_1) + \ldots + f(v_r)$ with $f(v_i) \in V_i$ and as $V = V_1 \oplus \ldots \oplus V_r$ is a direct sum, we get

$$f(v_1) + \ldots + f(v_r) = f(v) = \lambda v = \lambda v_1 + \cdots + \lambda v_r$$

if and only if $f(v_i) = \lambda v_i$ for all i.

Important. Here, we really need both the fact that the V_i's are f-invariant *and* that $V = V_1 \oplus \ldots \oplus V_r$ is a direct sum to justify

$$f(v_1 + \cdots + v_r) = f(v) = \lambda v \quad \Leftrightarrow \quad \forall i \in \{1, \ldots, r\} : \; f(v_i) = \lambda v_i.$$

Indeed the backward implication follows directly by factorising by λ but the forward implication is in general false if both conditions are not satisfied.

Hence we have

$$\mathrm{Eig}_\lambda(f) \;=\; \bigoplus_{i=1}^{r} \mathrm{Eig}_\lambda(f|_{V_i})$$

and thus

$$\dim \mathrm{Eig}_\lambda(f) \;=\; \sum_{i=1}^{r} \dim \mathrm{Eig}_\lambda(f|_{V_i}).$$

Therefore, the geometric multiplicity of λ as eigenvalue of f is equal to the sum over $1 \leqslant i \leqslant r$ of the geometric multiplicities of λ as eigenvalue of $f|_{V_i}$.

(b) By a theorem from the lecture an endomorphism of a finite dimensional vector space is diagonalizable if and only if its characteristic polynomial splits into linear factors and dor every of its eigenvalues the geometric multiplicity is equal to the arithmetic multiplicity.

The formula (1) yields that $\mathrm{char}_f(X)$ splits into linear factors if and only if for $\mathrm{char}_{f|_{V_i}}(X)$ does for every i. Consider an arbitrary eigenvalue $\lambda \in K$ of (f). Then we get from (a) and the fact, that the geometric multiplicity is always $\leqslant$ the arithmetic multiplicity, that these multiplicities are equal for f if and only if they are equal for every $f|_{V_i}$. Hence f is diagonalizable if and only if $f|_{V_i}$ is for all i.

(c) Assume that f and g are simultaneously diagonalizable. Then f and g are of course diagonalizable. Let $v_1, \ldots, v_n$ be a basis of V consisting of simultaneous eigenvectors of f and g with eigenvalues $\lambda_1, \ldots, \lambda_n$ for f and $\mu_1, \ldots, \mu_n$ for g. For every element $\sum_{i=1}^n \alpha_i v_i \in V$, we have

$$\begin{aligned} f\left(g\left(\sum_{i=1}^n \alpha_i v_i\right)\right) \;&=\; f\left(\sum_{i=1}^n \alpha_i \mu_i v_i\right) \;=\; \sum_{i=1}^n \alpha_i \mu_i \lambda_i v_i \\ &=\; \sum_{i=1}^n \alpha_i \lambda_i \mu_i v_i \;=\; g\left(\sum_{i=1}^n \alpha_i \lambda_i v_i\right) \;=\; g\left(f\left(\sum_{i=1}^n \alpha_i v_i\right)\right). \end{aligned}$$

Thus f and g commute.

Now assume that f and g commute and are both diagonalizable. As f is diagonalizable, there exist eigenvalues $\lambda_1, \ldots, \lambda_r$ of f with $V = \bigoplus_{i=1}^{r} \mathrm{Eig}_{\lambda_i}(f)$. For every $1 \leqslant i \leqslant r$ and $v \in \mathrm{Eig}_{\lambda_i}(f)$ commutativity of f and g yields:

$$f(g(v)) = g(f(v)) = g(\lambda_i v) = \lambda_i g(v)$$

and thus $g(v) \in \mathrm{Eig}_{\lambda_i}(f)$. The eigenspaces of f are hence g-invariant. As g is also diagonalizable, the map $g|_{\mathrm{Eig}_{\lambda_i}(f)}$ is also diagonalizable for every $1 \leqslant i \leqslant r$ according to (b). Thus there exists a basis B_i of $\mathrm{Eig}_{\lambda_i}(f)$ from eigenvectors of g. Together, we get that $B := B_1 \cup \cdots \cup B_r$ is a basis of V consisting of simultaneous eigenvectors of f and g; hence f and g are simultaneously diagonalizable.

6. Let K be a field and let V be an n-dimensional vector space over K ($n > 0$).

   (a) Let T be a diagonalizable endomorphism of V with (not necessarily distinct) eigenvalues λ_i for $1 \leqslant i \leqslant n$. Show that

   $$\mathrm{Tr}(T) = \sum_{i=1}^{n} \lambda_i \quad \text{and that} \quad \det(T) = \prod_{i=1}^{n} \lambda_i.$$

   For $0 \leqslant k \leqslant n$, let c_k be the coefficient of x^k in the characteristic polynomial of T. Give a formula for c_k in terms of the eigenvalues of T.

   (b) Let $B \in M_{2\times 2}(\mathbb{R})$ be diagonalizable with $\mathrm{Tr}(B) = 0$. Show that $\det(B) \leqslant 0$.

Solution:

(a) We know that we can factor the characteristic polynomial of T as

$$\mathrm{char}_T(X) = \prod_{i=1}^{n} (\lambda_i - X).$$

It follows from this formula that

$$c_k = (-1)^k \sum_{\substack{\{i_1, i_2, \ldots, i_{n-k}\} \in \{1, \ldots, n\}^{n-k} \\ i_r \neq i_s \text{ for } r \neq s}} \lambda_{i_1} \lambda_{i_2} \cdots \lambda_{i_{n-k}}$$

In particular,

$$c_0 = \prod_{i=1}^{n} \lambda_i$$

$$c_{n-1} = (-1)^{n-1} \sum_{i=1}^{n} \lambda_i.$$

You have seen in the lectures that $c_0 = \det(T)$ and $c_{n-1} = (-1)^{n-1}\operatorname{Tr}(T)$. This shows what we wanted.

Aliter: Since T is diagonalizable, by definition there exists a basis $\mathcal{B}$ of V such that

$$[T]_{\mathcal{B}}^{\mathcal{B}} = \begin{pmatrix} \lambda_1 & & & \\ & \lambda_2 & & \\ & & \ddots & \\ & & & \lambda_n \end{pmatrix}.$$

You have seen in the lectures that the trace and the determinant do not depend of the choice of basis, hence by a direct computation we have

$$\operatorname{Tr}(T) = \operatorname{Tr}([T]_{\mathcal{B}}^{\mathcal{B}}) = \sum_{i=1}^{n} \lambda_i \quad \text{and} \quad \det(T) = \det([T]_{\mathcal{B}}^{\mathcal{B}}) = \prod_{i=1}^{n} \lambda_i.$$

(b) Since B is diagonalizable, we can use (a) and observe that

$$\operatorname{Tr}(B) = 0 \iff \lambda_1 + \lambda_2 = 0 \iff \lambda_1 = -\lambda_2,$$

where λ_i denote the eigenvalues of B. Hence, using (a) again,

$$\det(B) = -\lambda_1^2 \leqslant 0$$

since $\lambda_1^2 \geqslant 0$.