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Musterlösung Serie 19

Scalar products, bilinear forms

1. For which values of $a \in \mathbb{R}$ does the expression

$$\langle x, y\rangle := x_1y_1 + ax_1y_2 + ax_2y_1 + 7x_2y_2,$$

where $x = \binom{x_1}{x_2}$ and $y = \binom{y_1}{y_2}$, define an inner product on $\mathbb{R}^2$?

Lösung: We check directly that $\langle\cdot,\cdot\rangle$ is a symmetric bilinear form on $\mathbb{R}^2$; i.e. by considering the studying the symmetric representation matrix $\left(\begin{smallmatrix}1 & a\\ a & 7\end{smallmatrix}\right)$. We compute

$$\begin{aligned}\langle x, x\rangle &= x_1^2 + 2ax_1x_2 + 7x_2^2\\ &= x_1^2 + 2ax_1x_2 + a^2x_2^2 - a^2x_2^2 + 7x_2^2\\ &= (x_1 + ax_2)^2 + (7 - a^2)x_2^2.\end{aligned}$$

If $\langle\cdot,\cdot\rangle$ is positive definite, we get with $x = \binom{-a}{1} \neq 0$, that $7 - a^2 > 0$ holds; hence $|a| < \sqrt{7}$. Conversely, if we have $|a| < \sqrt{7}$, we get from the upper computation $\langle x, x\rangle > 0$ for all $x \neq 0$ and hence $\langle\cdot,\cdot\rangle$ is positive definite. Hence $\langle\cdot,\cdot\rangle$ is an inner product if and only if $|a| < \sqrt{7}$.

2. Let V be the vector space of real polynomials of degree at most n.

   (a) Show that the expression

$$\langle p, q\rangle := \int_0^\infty p(t)q(t)e^{-t}\,dt$$

   defines an inner product on V.

   (b) Find the matrix of the inner product with respect to the basis $1, x, \dots, x^n$.

Lösung:

   (a) We know from analysis that these improper integral converge. Linearity of the integral yields that $\langle\cdot,\cdot\rangle$ is a bilinear form, which is obviously symmetric. Now let $p \in V \smallsetminus \{0\}$ be arbitrary. Choose a point $x_0 > 0$ with $p(x_0) \neq 0$. As p induces a continuous function, we get $|p(x)| \geqslant c := \frac{1}{2}|p(x_0)| > 0$ on an interval $[a, b] \subset \mathbb{R}^{\geqslant 0}$ with $x_0 \in [a, b]$. As the function $t \mapsto e^{-t}$ is strictly decreasing and $e^{-t} > 0$ holds for all t, we get

$$\langle p, p\rangle \;=\; \int_0^\infty p(t)^2e^{-t}\,dt \;\geqslant\; \int_a^b p(t)^2e^{-t}\,dt \;\geqslant\; c\cdot e^{-b}\cdot(b-a) \;>\; 0.$$

   Hence $\langle\cdot,\cdot\rangle$ is positive definite and thus a scalar product.

(b) For all $k \in \mathbb{Z}^{\geqslant 0}$ consider

$$a(k) := \int_0^\infty t^k e^{-t}\,dt.$$

Then

$$a(0) = \int_0^\infty e^{-t}\,dt = -e^{-t}\Big|_0^\infty = 1,$$

and for $k \geqslant 1$ partial integration yields

$$a(k) = \int_0^\infty t^k e^{-t}\,dt = -t^k e^{-t}\Big|_0^\infty - \int_0^\infty k t^{k-1}(-e^{-t})\,dt = k \cdot a(k-1).$$

By induction over k we get $a(k) = k!$. Now let $A := (a_{ij})_{i,j}$ be the representation matrix of the scalar product $\langle \cdot, \cdot \rangle$ with respect to the ordered basis $(1, x, \ldots, x^n)$. Then

$$a_{ij} = \langle x^{i-1}, x^{j-1} \rangle = a(i+j-2) = (i+j-2)!\,.$$

3. Let $V = \mathbb{R}^2$ endowed with the standard scalar product, and for $i = 1, 2$, let $v_i \in V \setminus \{0\}$. Show that the formula

$$\langle v_1, v_2 \rangle = ||v_1||\,||v_2|| \cos(\widehat{v_1, v_2}),$$

defining the cosinus of an angle is rotation-invariant. In other words show that, for any rotation of the plane $R : V \to V$, we have

$$\cos(\widehat{v_1, v_2}) = \cos(\widehat{Rv_1, Rv_2}),$$

which is what we would expect from a good definition of the angle between 2 vectors.

Solution: Let v_1, $v_2 \in V$ be pair of non-vanishing vectors, and let $\theta \in [0, 2\pi)$. We show that the rotation matrix

$$\begin{pmatrix} \cos(\theta) & \sin(\theta) \\ -\sin(\theta) & \cos(\theta) \end{pmatrix}$$

rotating the plane by θ preserves the scalar product, and therefore also leaves the norm unchanged. Indeed, for any pair of vectors $v, w \in V$, we have

$$\begin{aligned}
&\left\langle \begin{pmatrix} \cos(\theta) & \sin(\theta) \\ -\sin(\theta) & \cos(\theta) \end{pmatrix} v, \begin{pmatrix} \cos(\theta) & \sin(\theta) \\ -\sin(\theta) & \cos(\theta) \end{pmatrix} w \right\rangle \\
=& \left(\begin{pmatrix} \cos(\theta) & \sin(\theta) \\ -\sin(\theta) & \cos(\theta) \end{pmatrix} \cdot v \right)^T \cdot \begin{pmatrix} \cos(\theta) & \sin(\theta) \\ -\sin(\theta) & \cos(\theta) \end{pmatrix} \cdot w \\
=& v^T \cdot \begin{pmatrix} \cos(\theta) & \sin(\theta) \\ -\sin(\theta) & \cos(\theta) \end{pmatrix}^T \cdot \begin{pmatrix} \cos(\theta) & \sin(\theta) \\ -\sin(\theta) & \cos(\theta) \end{pmatrix} \cdot w \\
=& v^T \cdot I_2 \cdot w \\
=& \langle v, w \rangle\,.
\end{aligned}$$

This shows

$$\frac{\langle v_1, v_2\rangle}{||v_1||\,||v_2||} = \frac{\langle Rv_1, Rv_2\rangle}{||Rv_1||\,||Rv_2||} \implies \cos(\widehat{v_1, v_2}) = \cos(\widehat{Rv_1, Rv_2}).$$

4. Let $A \in \mathrm{Mat}_{n\times n}(\mathbb{R})$. Show that:
   (a) The matrix $A^T A$ is symmetric.
   (b) The matrix $A^T A$ is positive-definite if and only if A is invertible.
   (c) It holds that $\mathrm{Rang}(A^T A) = \mathrm{Rang}(A)$.

Lösung: Assertion (a) follows from $(A^T A)^T = A^T (A^T)^T = A^T A$.

As a preparation, we compute for every $v \in \mathbb{R}^n$

$$(*) \qquad v^T \cdot (A^T A) \cdot v \;=\; (v^T A^T) \cdot Av \;=\; (Av)^T Av \;=\; \|Av\|^2,$$

where $||\ ||$ is the standard euclidean norm on $\mathbb{R}^n$.

If A is invertible, it follows that $Av \neq 0$ and thus $v^T \cdot (A^T A) \cdot v = \|Av\|^2 > 0$. Therefore, the matrix $A^T A$ is positive definite. Conversely, if $A^T A$ is positive definite, we get $\|Av\|^2 = v^T \cdot (A^T A) \cdot v > 0$ and hence $Av \neq 0$. Thus the linear map $L_A\colon \mathbb{R}^n \to \mathbb{R}^n$, $v \mapsto Av$ has trivial kernel and hence A is invertible. So (b) is proved.

For (c) we start by claiming $\mathrm{kernel}(L_A) = \mathrm{kernel}(L_{A^T A})$.

Proof: For all$v \in \mathbb{R}^n$ with $Av = 0$, we have $A^T Av = A^T 0 = 0$; i.e. „$\subset$" holds. Conversely, let $A^T Av = 0$. From $(*)$ we then get $\|Av\|^2 = v^T \cdot (A^T A) \cdot v = v^T 0 = 0$. As the euclidean norm is positive definite $Av = 0$, also $v \in \mathrm{Kern}(L_A)$. This shows the inclusion „$\supset$", and the claim is proved. *q.e.d.*

The claim yields

$$\begin{aligned}\mathrm{rank}(A) \;&=\; \dim \mathrm{Im}(L_A) \;=\; n - \dim\ker(L_A)\\ &=\; n - \dim\ker(L_{A^T A}) \;=\; \dim\mathrm{Im}(L_{A^T A}) \;=\; \mathrm{rank}(A^T A).\end{aligned}$$

5. (a) Let $\|\cdot\|$ be a norm on the $\mathbb{R}$-vector space V. Show that the norm is induced by an inner product $\langle\cdot,\cdot\rangle$ on V if and only if it satisfies the *parallelogram identity*

   $$\|x+y\|^2 + \|x-y\|^2 = 2\|x\|^2 + 2\|y\|^2$$

   for all $x, y \in V$.

   (b) Let V be a finite-dimensional $\mathbb{R}$ vector space. Consider the following map:

   $$\begin{array}{rccc} ||\cdot||_1 : & V & \to & \mathbb{R}_{\geqslant 0} \\ & v = (v_1, v_2, \ldots, v_n) & \mapsto & \sum_{i=1}^n |v_i| \end{array}$$

   Check that $||\cdot||_1$ defines a norm on V and prove that it does not come from a scalar product.

Lösung:

(a) If $||x||^2 = \langle x, x\rangle$, we have

$$\begin{aligned} ||x+y||^2 &= \langle x+y, x+y\rangle \\ &= \langle x,x\rangle + 2\langle x,y\rangle + \langle y,y\rangle \\ &= ||x||^2 + 2\langle x,y\rangle + ||y||^2 \\ ||x-y||^2 &= \langle x-y, x-y\rangle \\ &= \langle x,x\rangle - 2\langle x,y\rangle + \langle y,y\rangle \\ &= ||x||^2 - 2\langle x,y\rangle + ||y||^2, \end{aligned}$$

hence

$$||x+y||^2 + ||x-y||^2 = 2\,||x||^2 + 2\,||y||^2 .$$

Conversely, let $||\cdot||$ be a norm on V satisfying the parallelogram identity(PI). Define

$$\langle x,y\rangle := \frac{1}{2}\big(||x+y||^2 - ||x||^2 - ||y||^2\big) = \frac{1}{4}\big(||x+y||^2 - ||x-y||^2\big),$$

where the last equation follows from (PI). This map satisfies $\langle x,y\rangle = \langle y,x\rangle$ and $||x||^2 = \langle x,x\rangle$ for all $x,y \in V$. As $||\cdot||$ is already positive definite, it remains to show that $\langle\cdot,\cdot\rangle$ is bilinear. By symmetry, it is enough to show that the map is linear in the first variable.
For arbitrary $x, x', y \in V$ we compute

$$\begin{aligned} \langle x+x', y\rangle &= \frac{1}{4}(||(x+y)+x'||^2 - ||x+x'-y||^2) \\ &\overset{(PI)}{=} \frac{1}{4}(2\,||x+y||^2 + 2\,||x'||^2 - ||x+y-x'||^2 - ||x+x'-y||^2) \\ &\overset{(PI)}{=} \frac{1}{4}(2\,||x+y||^2 + 2\,||x'||^2 - 2\,||x||^2 - 2\,||x'-y||^2) \\ &= \frac{1}{4}(2\,||x+y||^2 - 2\,||x||^2 - 2\,||x'||^2 + 4\,||x'||^2 - 2\,||x'-y||^2) \\ &\overset{(PI)}{=} \frac{1}{4}(2\,||x+y||^2 - 2\,||x||^2 - 2\,||x'||^2 + 2\,||x'+y||^2 - 4\,||y||^2) \\ &= \frac{1}{2}(||x+y||^2 - ||x||^2 - ||y||^2 + ||x'+y||^2 - ||x'||^2 - ||y||^2) \\ &= \langle x,y\rangle + \langle x',y\rangle. \end{aligned}$$

Hence the map is additive in the first variable. We study its behaviour under scalar multiplication in an inderect way. First, we prove the following assertions for all $n \in \mathbb{Z}^{>0}$ and all $x,y \in V$:

(i) From

$$0 = \langle 0,y\rangle = \langle x+(-x),y\rangle = \langle x,y\rangle + \langle -x,y\rangle$$

we get $\langle -x,y\rangle = -\langle x,y\rangle$.

(ii) Additivity yields by induction $\langle nx, y\rangle = n\langle x, y\rangle$.

(iii) From (ii) with $\frac{1}{n}x$ instead of x we get

$$\langle x, y\rangle = \langle n\frac{1}{n}x, y\rangle = n\langle\frac{1}{n}x, y\rangle,$$

and hence $\langle\frac{1}{n}x, y\rangle = \frac{1}{n}\langle x, y\rangle$.

All three assertions together now yield

$$\langle\frac{p}{q}\cdot x, y\rangle = \frac{p}{q}\cdot\langle x, y\rangle$$

for all $\frac{p}{q}\in\mathbb{Q}$ and all $x, y\in V$.

Now fix arbitrary $x, y\in V$. The subspace U spanned by this elements is then is isomorphic to$\mathbb{R}^n$ for $n\leqslant 2$. The restriction of $||\cdot||$ to U is again a norm and thus corresponds to a norm on $\mathbb{R}^n$. In the lecture we saw that this norm is a Lipschitz-continuous function on $\mathbb{R}^n$. In particular, it is continuous. Hence, the restriction of $\langle\cdot,\cdot\rangle$ on $U\times U$ corresponds to a continuous function on $\mathbb{R}^n\times\mathbb{R}^n$. This implies the continuity of

$$\mathbb{R}\to\mathbb{R},\ t\mapsto\langle tx, y\rangle - t\langle x, y\rangle.$$

The upper equation shows, that this function vanishes on $\mathbb{Q}$. As it is continuous, it vanishes on $\mathbb{R}$. Hence, we get

$$\langle tx, y\rangle = t\langle x, y\rangle$$

for all $t\in\mathbb{R}$. Therefore $\langle\cdot,\cdot\rangle$ is linear in the first variable, and we are done.

(b) To show that $||\cdot||_1$ is not induced by an scalar product, we just need to find a pair a vectors for which $||\cdot||_1$ does not satisfy the parallelogram identity. Let $x = (1, 0, 0, \ldots, 0)^T$, $y = (0, 1, 0, \ldots, 0)^T$. We have

$$\begin{aligned} ||x+y||_1^2 &= (1+1)^2 = 4\\ ||x-y||_1^2 &= (1+1)^2 = 4\\ 2\,||x||_1^2 &= 2\\ 2\,||y||_1^2 &= 2. \end{aligned}$$

So the parallelogram identity is not satisfied.

6. Let $K=\mathbb{R}$, $V = M_{n\times n}(K)$, and consider the map

$$\begin{array}{ccc} V\times V & \to & K\\ (A,B) & \mapsto & \operatorname{Tr}(A^TB). \end{array}$$

Show that it defines an inner product on V and find an orthonormal basis with respect to this inner product. The induced norm is called the *Hilbert-Schmidt norm*. Give a formula form the norm of a matrix $A \in V$ in terms of its entries.

Solution: The basic rules of matrix multiplication imply that the map $(A, B) \mapsto A^T B$ is bilinear. As the trace map is linear, the given map is bilinear. From

$$\langle A, B\rangle = \mathrm{Tr}(A^T B) = \mathrm{Tr}((A^T B)^T) = \mathrm{Tr}(B^T (A^T)^T) = \mathrm{Tr}(B^T A) = \langle B, A\rangle$$

we get that it is also symmetric. Now write $A = (v_1, \ldots, v_n)$ with column vectors v_i. Then A^T is the matrix with rows $v_1^T, \ldots, v_n^T$ and thus

$$A^T A \;=\; \left(v_i^T v_j\right)_{i,j=1,..,n}.$$

The trace is defined as sum of the diagonal entries; hence

$$\langle A, A\rangle \;=\; \mathrm{Tr}(A^T A) \;=\; \sum_{i=1}^{n} v_i^T v_i.$$

Here every summand $v_i^T v_i$ is the square of the absolute value of v_i with respect to the standard scalar product on $\mathbb{R}^n$ and hence $\geqslant 0$. Thus $\langle A, A\rangle \geqslant 0$. For $A \neq 0$ at least for one i we have $v_i \neq 0$, so at least one summand is > 0 and therefore $\langle A, A\rangle > 0$. In summary, we showed that $\langle \cdot, \cdot\rangle$ is a scalar product.

The set of all $n \times n$-elementary matrices forms a basis of $\mathrm{Mat}_{n\times n}(\mathbb{R})$. Direct computations show

$$\langle E_{ij}, E_{k\ell}\rangle = \begin{cases} 1 & \text{if } (i,j) = (k,\ell), \\ 0 & \text{else,} \end{cases}$$

hence this is an orthonormal basis

Aliter: We identify $\mathrm{Mat}_{n\times n}(\mathbb{R})$ with $\mathbb{R}^{n^2}$, by listing the coefficients of a matrix in a fixed order. This is an isomorphism of vector spaces. For two $n \times n$-matrices $A = (a_{ij})_{i,j}$ and $B = (b_{ij})_{i,j}$ a direct computation shows

$$\langle A, B\rangle = \sum_{i=1}^{n}\sum_{j=1}^{n} a_{ij} b_{ij}.$$

Under the mentioned isomorphism $\langle \cdot, \cdot\rangle$ corresponds to the standard scalar product on $\mathbb{R}^{n^2}$; and hence this is also a scalar product. Moreover, the $n \times n$-elementary matrices correspond exactly to the standard basis vectors of $\mathbb{R}^{n^2}$; and as these form an orthonormal basis, the same holds true for $n \times n$-elementary matrices.

We denote the induced norm $||\cdot||$. Let us denote $A = (a_{ij})_{1\leqslant i,j\leqslant n}$, $A^T = (\tilde{a}_{ij})_{1\leqslant i,j\leqslant n}$, $A^T A = (c_{ij})_{1\leqslant i,j\leqslant n}$. We have

$$c_{ij} = \sum_{k=1} \tilde{a}_{ik} a_{kj} = \sum_{k=1} a_{ki} a_{kj}.$$

Hence $c_{ii} = \sum_{k=1}^{n} a_{ki}^2$ and

$$||A||^2 = \mathrm{Tr}(A^T A) = \sum_{i=1}^{n} c_{ii} = \sum_{i=1}^{n} \sum_{k=1}^{n} a_{ki}^2$$

is simply the Euclidian norm of A viewed as an element of $\mathbb{R}^{n^2}$.